I. Heine-Borel Theorem

Recap

- Defn X compact if every open cover of X has a finite subcover
- Boundedness Thm: X compact, $f : X \to \mathbb{R}$ cts $\Rightarrow$ f bounded
- Heine-Borel Thm: $X \subset \mathbb{R}^n$ compact $\Longleftrightarrow$ X closed and bounded

$(\Rightarrow)$ last time. $(\Leftarrow)$ today

Prop X compact, $A \subset X$ closed $\Rightarrow$ A compact

Thm $[0,1]^n \subset \mathbb{R}^n$ is compact

(Hence also $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ compact.)

Prof of Heine-Borel. Assume $X \subset \mathbb{R}^n$ closed & bounded.

X bounded $\Rightarrow$ $X \subset Q$ some closed rectangle

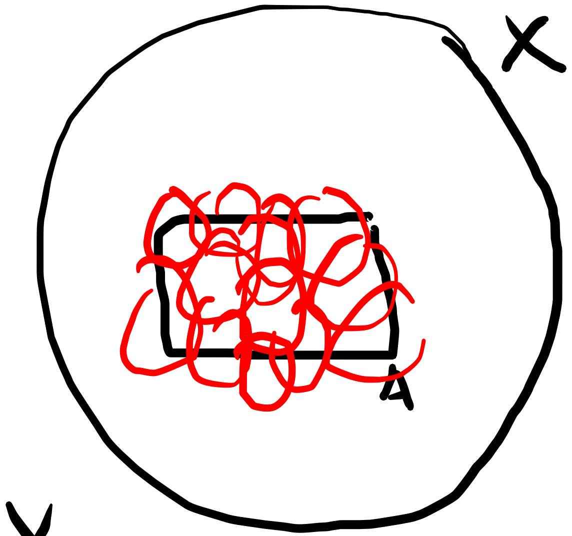
Q compact, X closed $\Rightarrow$ X compact. (by prop)
(by Thm)

$\square$

Prop X compact, $A \subset X$ closed $\Rightarrow$ A compact

Proof Let $\mathcal{U}$ be any open cover of A.

Then $\mathcal{U} \cup \{X \setminus A\}$ open cover of X.



X compact $\Rightarrow$ there is a finite subcover

$\{U_1, \ldots, U_m\} \cup \{X \setminus A\}$ of X

$\Rightarrow$ $\{U_1, \ldots, U_m\}$ is a cover of A. $\square$

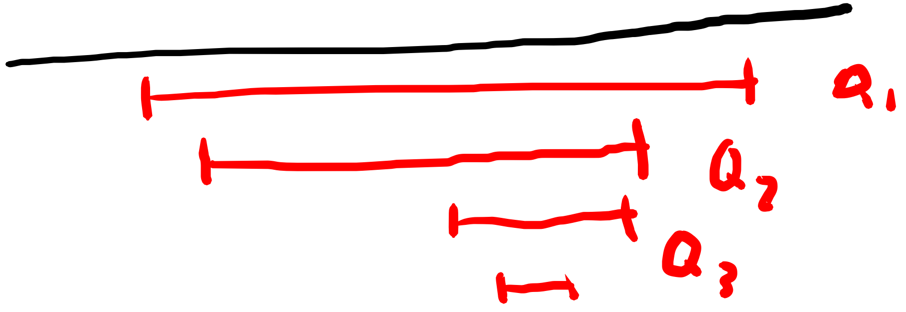
II. Compactness of $[0,1]^n$

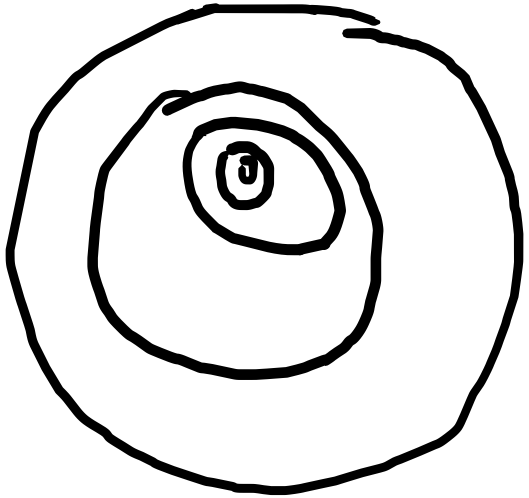
Main ingredient

Thm (onion ring / nested interval)

$Q_i \subset \mathbb{R}^n$ nested closed rectangles $Q_{i+1} \subset Q_i$.

Then $\bigcap Q_i \neq \phi$





Thm follows from case $n=1$.

case $n=1$ follow from the least upper bound property for $\mathbb{R}$.

Least upper bound property on $\mathbb{R}$:

every nonempty subset $A \subset \mathbb{R}$ that's bounded above has a least upper bound.

<u>Defn</u> $A \subset \mathbb{R}$ <u>bounded above</u> if $\exists$ $b \in \mathbb{R}$ st. $a \leq b$ $\forall a \in A$.

Say b is an <u>upper bound</u>.

Say b is a <u>least upper bound</u> if b' is another upper bound then $b \leq b'$.

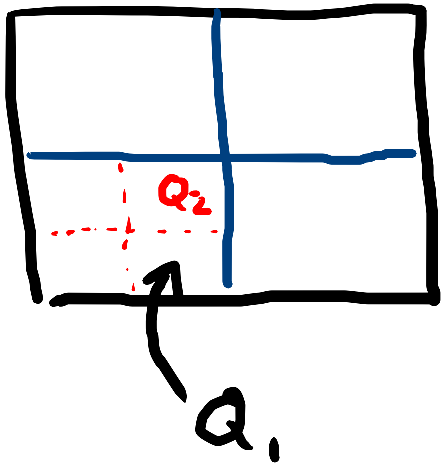
eg. $A = \mathbb{Z} \subset \mathbb{R}$ not bounded

$A = [0,1]$ bounded above LUB $= 1$.

Proof that $[0,1]^n$ compact (method of bisection)

By contradiction: suppose $\exists$ open cover $\mathcal{U}$ with no finite subcover

Divide $[0,1]^n$ into quadrants. By assumption one of the quadrants Q_1 not covered by finitely many elements of $\mathcal{U}$.



Repeat to get nested closed rectangles $Q_{i+1} \subset Q_i$ s.t.

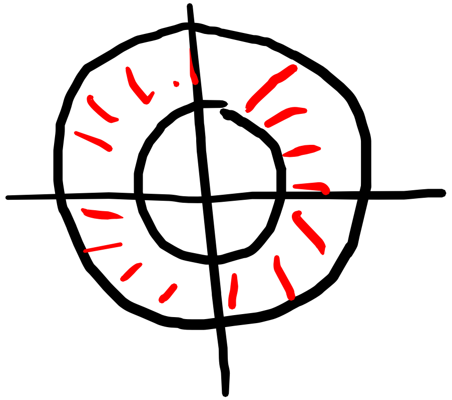
(1) Q_i not covered by finite subset of $\mathcal{U}$.

(2) side lengths of Q_i are $\frac{1}{2^i}$

onion ring theorem $\Rightarrow \exists\, z \in \bigcap Q_i$. Take $U \in \mathcal{U}$ w/ $z \in U$. U open, $Q_i \subset U$ for $i >> 0$. (by (2)). This contradicts (1). $\square$

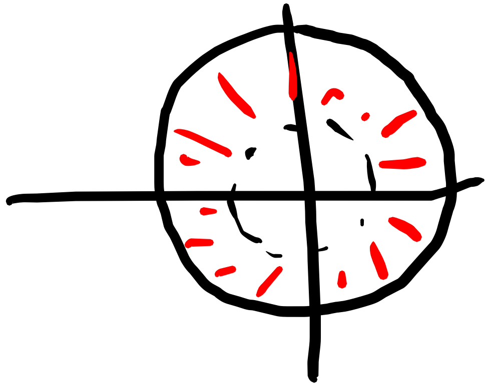
Ex

- Annulus $A = \{ z \in \mathbb{R}^2 : 1 \le |z| \le 4 \}$



A compact b/c closed & bounded.

- $B = \{ z \in \mathbb{R}^2 : 1 < |z| \le 4 \}$

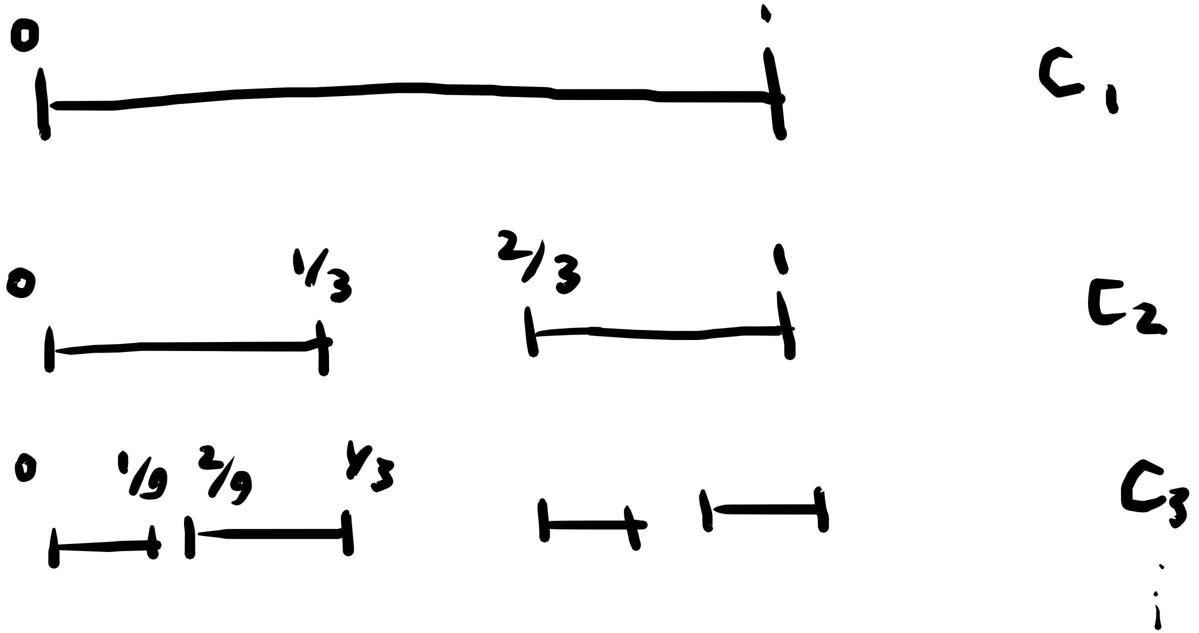


not compact

$$f : B \longrightarrow \mathbb{R}$$
$$z \longmapsto \frac{1}{|z| - 1}$$

continuous, unbounded.

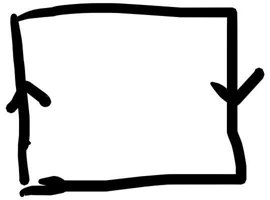
- Cantor set



$C = \bigcap C_i$ closed (intersection of closed) $\Rightarrow$ compact.

bound ✓

- Mobius band is compact

$M =$ 

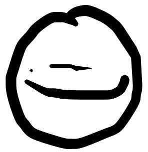
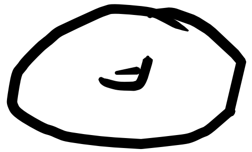
There is a gluing map $f: [0,1]^2 \longrightarrow M$ which is continuous (need to topologize M quotient topology)

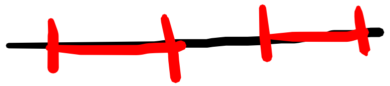
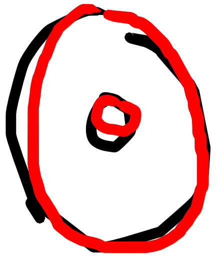
$[0,1]^2$ compact $\Rightarrow f([0,1]^2) = M$ compact.

III. Connectedness

topological invariant, captures "pieces/components" of a space.

intuition:

connected $\mathbb{R}$, $\mathbb{R}^n$, $[0,1]$, , 

disconnected , cantor set, ∂ annulus 

Defn X disconnected if $\exists$ nonempty open U, V s.t.

$X = U \cup V$ and $U \cap V = \emptyset$

Connectedness examples

Q connected?

$$GL_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}.$$

topologist sine curve