I. Surfaces and their classification

"Topology is the study of topological spaces and their invariants (like Euler number)"

<u>Defn</u> A surface is a topological space S s.t.

(1) $\forall x \in S$ $\exists$ open $x \in U$ and a topological equivalence

$$U \longrightarrow R^2$$

(2) S is Hausdorff (for $x, y \in S$ $\exists$ open $x \in U$, $y \in V$ s.t. $U \cap V = \phi$).

<u>Rmk</u> Hausdorff is a technical condition but a reasonable one:

eg

Ex (plane with two origins)

$X = \mathbb{R}^2 \cup \{p\}$ with topology

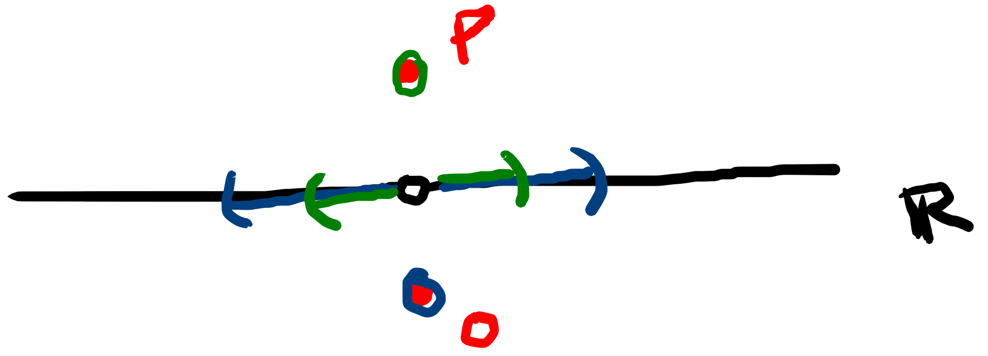
open sets are
- open subsets of $\mathbb{R}^2$
- set $U = \{p\} \cup (V \setminus \{0\})$

where $V \subset \mathbb{R}^2$ open set containing 0.

Note X satisfies (1)

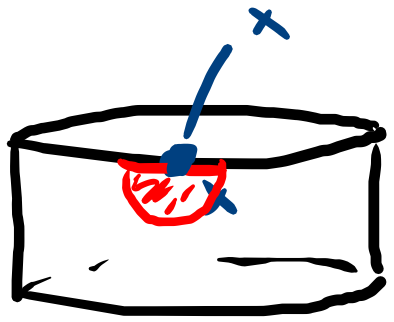
but X is not Hausdorff b/c if $U_1 \ni 0$ and $U_2 \ni p$ open

then $U_1 \cap U_2 \neq \emptyset$.

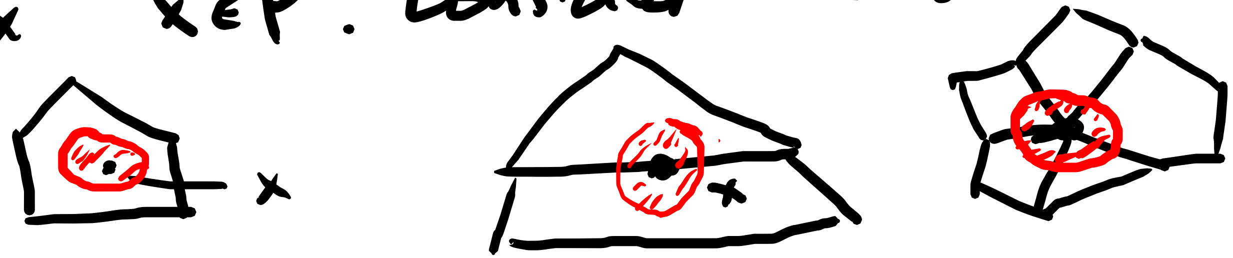


Examples

- $\mathbb{R}^2$ is a surface. So is $\mathbb{R}^2 \setminus$ finite set
- $\mathbb{R}$ is not a surface because an interval
  $(a,b) \cong \mathbb{R}$ is not top. equivalent to $\mathbb{R}^2$ (later)
- Annulus $S^1 \times [0,1]$ not a surface
  but it is a "surface with boundary"

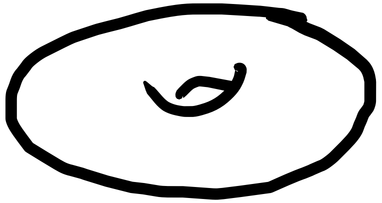
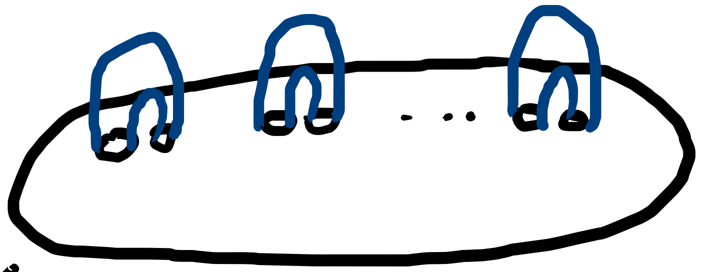


- A polyhedron P is a surface
  Fix $x \in P$. Consider case:
  - $x \in$ interior of a face
  - $x \in$ interior of edge
  - x is a vertex

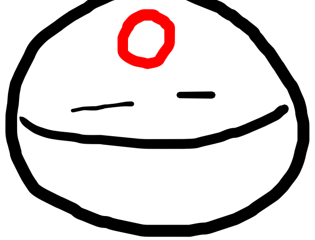


Surface constructions (surgery)

-  take S^2 and remove 2 disks and glue an annulus along the boundary.

$\rightsquigarrow$  

do this operation g times. Call result M_g

-   remove disk from S^2 and glue möbius band along boundary

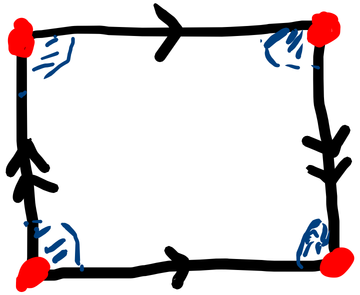
do this k times. Call result N_k.

<u>Thm</u> (classification of surfaces)

- Every (compact, connected) surface is topologically equivalent to one of M_g, N_k $\quad g \geq 0$, $k \geq 1$

- No two of these are top. equivalent.

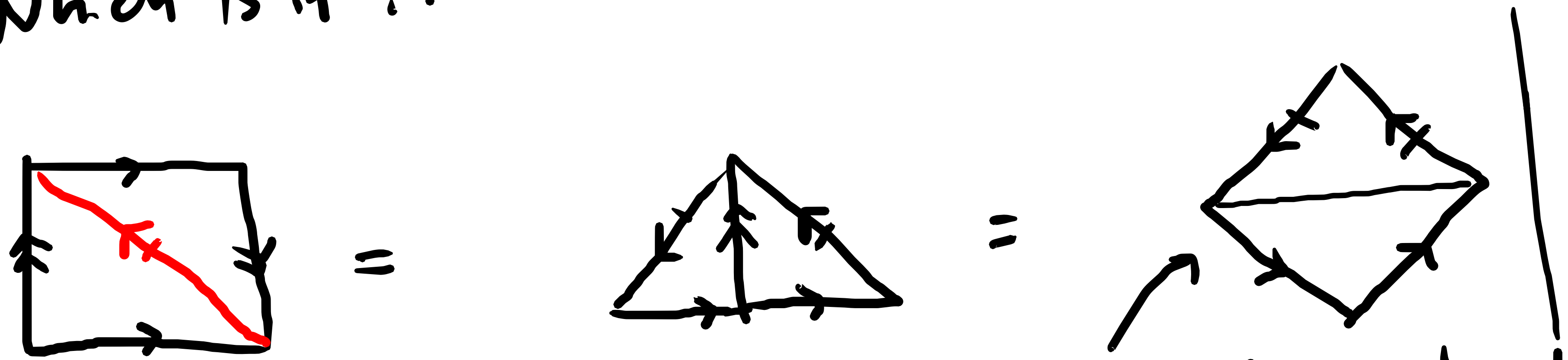
Ex The Klein bottle K is a surface



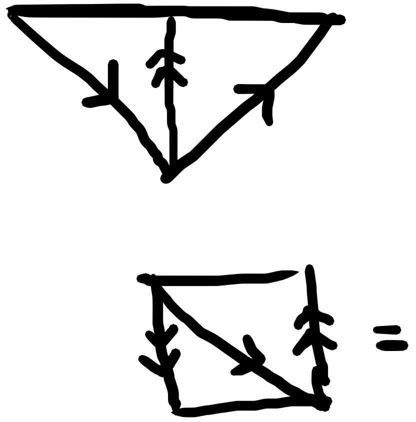
By classification

$K \cong$ either M_g or N_k for some g or k.

Which is it??



2 Möbius bands



Möbius

$K \cong$ two Mobius bands glued to an annulus. $\cong N_2$

Thm (classification of surfaces)

- Every (compact, connected) surface is topologically equivalent to one of M_g, N_k $g \geq 0$, $k \geq 1$

- No two of these are top. equivalent.

Rmk
- Showing $X \cong Y$ is a constructive problem (compare w/ Klein bottle above or proof that $S^2 \setminus p \cong \mathbb{R}^2$)

- Showing $X \not\cong Y$ is an "obstructive problem":

Thm 1 $\mathbb{R} \not\cong \mathbb{Z}$ (give $\mathbb{Z}$ topology as subspace of $\mathbb{R}$ $\to$ discrete topology)

Thm 2 $\mathbb{R} \not\cong \mathbb{R}^2$

Proof of Thm 1:

Observation: if $X \cong Y$ then X, Y have same cardinality (b/c a top equiv $f: X \to Y$ is in particular a bijection)

But $\mathbb{R}, \mathbb{Z}$ not same cardinality

$\Rightarrow \mathbb{R} \not\cong \mathbb{Z}$. $\square$

<u>Thm 1</u> $\mathbb{R} \not\cong \mathbb{Z}$ (give $\mathbb{Z}$ topology as subspace of $\mathbb{R} \to$ discrete topology)

<u>Thm 2</u> $\mathbb{R} \not\cong \mathbb{R}^2$

<u>Note</u> $\mathbb{R}, \mathbb{R}^2$ have same cardinality

How to show $\mathbb{R} \not\cong \mathbb{R}^2$. Can't check every bijection $f: \mathbb{R} \to \mathbb{R}^2$

Instead: find $\underbrace{\text{property}}_{\text{invariant}}$ that these spaces don't share

<u>Proof of Thm 2</u>

Thm 2 $\quad \mathbb{R} \not\cong \mathbb{R}^2$

Proof of Thm 2

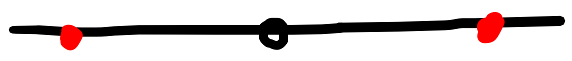
- Observation: if $\mathbb{R} \cong \mathbb{R}^2$ then also $\mathbb{R} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\}$.

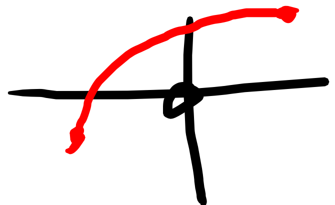
  $\mathbb{R} \xrightarrow{f} \mathbb{R}^2$ wlog $f(0) = 0$ so f restricts to

  $f| : \mathbb{R}\setminus 0 \longrightarrow \mathbb{R}^2 \setminus 0$ top. equiv.

- observation: if $X \cong Y$ then X, Y have same number of path components

- $\mathbb{R}\setminus 0$ has two components but $\mathbb{R}^2 \setminus 0$ has one





$\Rightarrow \mathbb{R}\setminus 0 \not\cong \mathbb{R}^2 \setminus 0$

$\Rightarrow \mathbb{R} \not\cong \mathbb{R}^2$ $\square$

<u>Rmk</u> A property of a space (eg cardinality, # components, Euler number) that is preserved under $\cong$ is called a <u>topological invariant</u>

To show $X \not\cong Y$, find an invariant that takes different values on X & Y