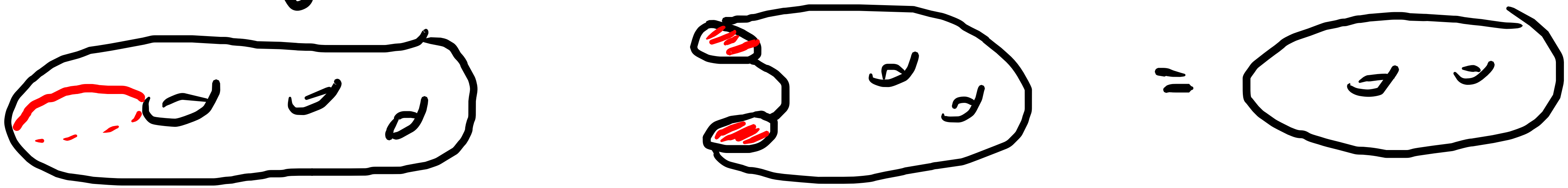
I. Classification of Surfaces proof Outline

(CoS) Every closed surface is obtained from S^2 by annulus and Möbius attachments.

<u>Proof Outline:</u> (1) Surface can be triangulated.

(2) surgery ("inverse" to A- or M-attachment)



produces a "simpler" surface (χ increases)

(3) For any triangulated surface $S = |K|$

$$\chi(K) = V - E + F \le 2$$

with equality $\iff$ $|K| \cong S^2$

Conclude that given $S = |K|$ after finitely many surgeries get S^2 w/ finite set of marked disks.

Reversing surgery describes S as obtained from A & M - attachments. $\square$

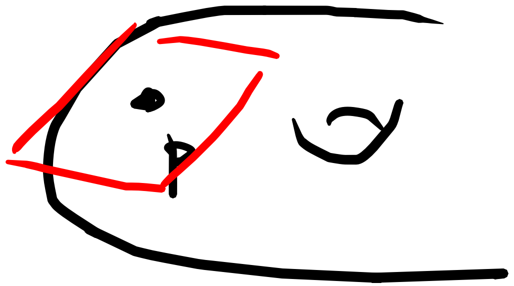
II. Triangulating Surfaces

Thm Any surface has a triangulation.

Proof sketch in special case: Assume $S \subset \mathbb{R}^3$ is "smooth"

near each $p \in S$, S locally looks like

linear subspace $\mathbb{R}^2 \subset \mathbb{R}^3$.

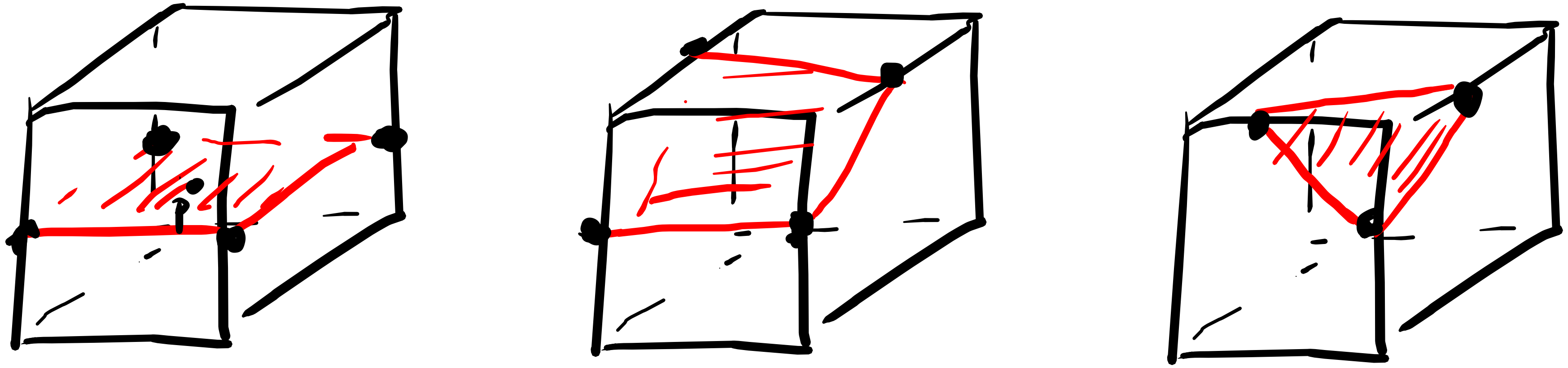


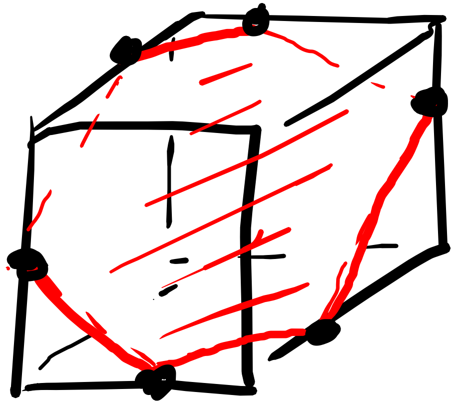
Given p can choose $0 < \varepsilon \ll 1$

s.t. if $C_\varepsilon(p)$ cube side length ε around p

then $S \cap C_\varepsilon(p) \approx$ planar cross section.

Sample cross sections

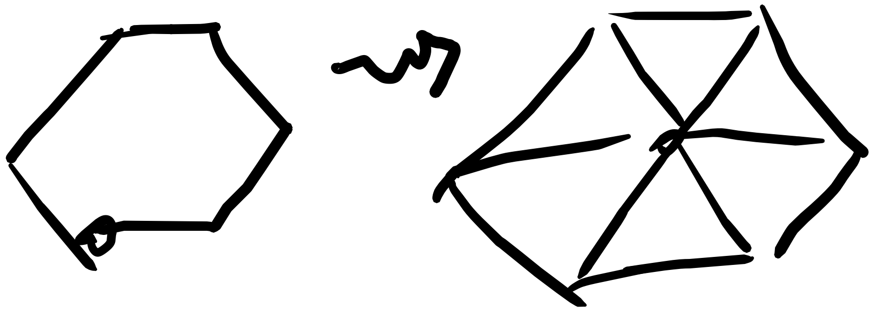




S compact $\Rightarrow$ choose ε "works for all $p \in S$"

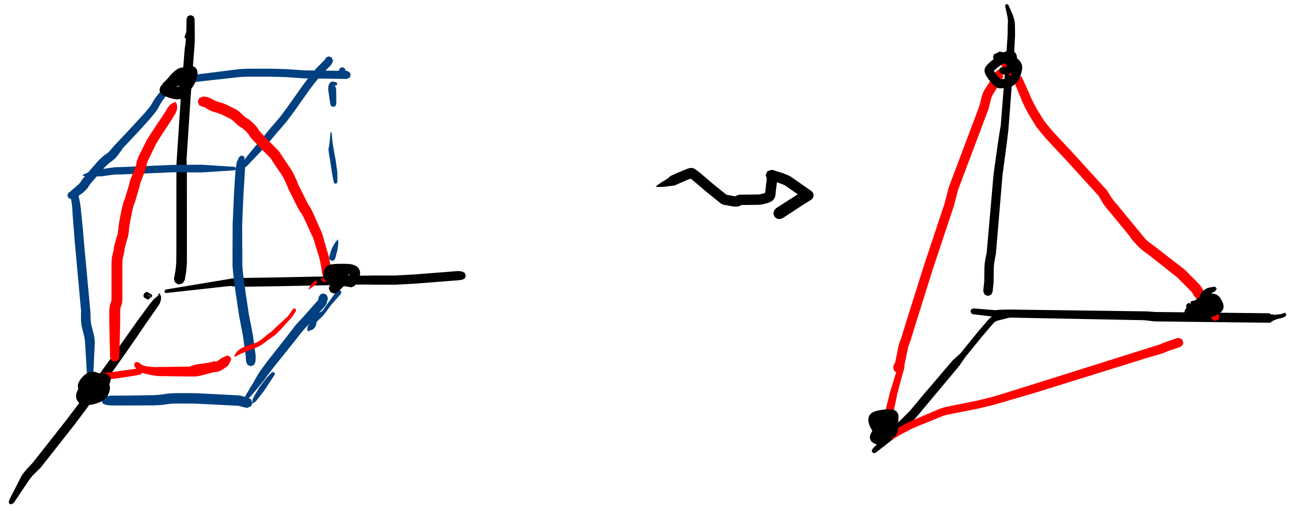
Tile $\mathbb{R}^3$ by cubes of length ε.

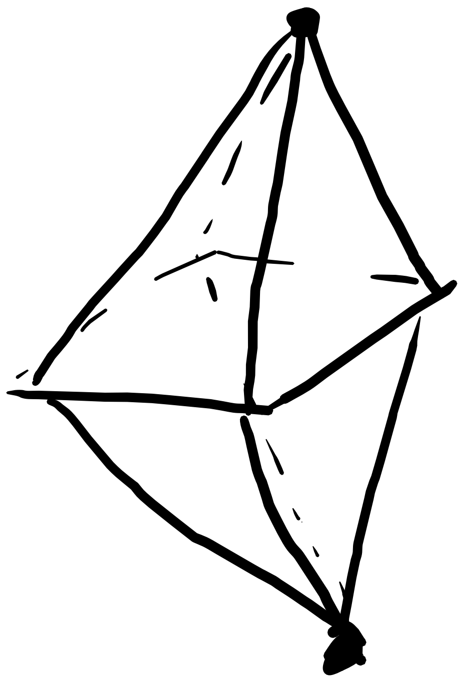
$\rightsquigarrow$ divides S into polygons



Further divide each polygon into triangles $\square$

Ex. $S^2 \subset \mathbb{R}^3$ use unit cubes





octahedron.

Rmks • Not every surface embeds $S \subset \mathbb{R}^3$.

eg $\mathbb{R}P^2$, K. but they do embed in $\mathbb{R}^4$ (whitney embedding Diff Top.)

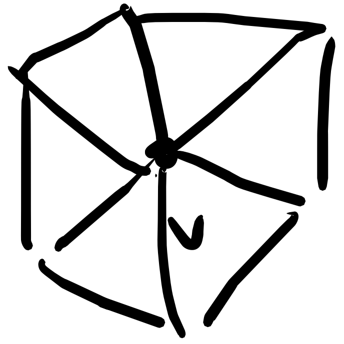
• Every surface "has a smooth structure"

Properties of the triangulation:

(1) 2-dimensional (no k-simplices $k \geq 3$)

(2) every edge meets exactly 2 faces

(3) every vertex "has a pizza neighborhood"



Conversely for a triangulation K w/ these properties, $|K|$ is a surface.

$S = |K|$

Combinatorial surface = Surface + triangulation

III. Euler number

for L simplicial complex

$$\chi(L) = \sum_{n \geq 0} (-1)^n \, \#\{n\text{-simplices}\}$$

$|K|$ combinatorial surface $\quad \chi(L) = V - E + F.$

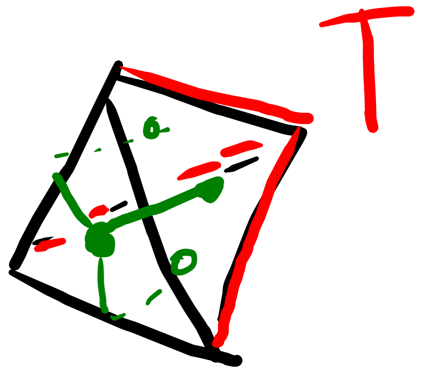
Lemma $|K|$ comb. surf

$\Rightarrow \chi(K) \leq 2$ (next time $\chi(K)=2$ $\Leftrightarrow |K| \cong S^2$)

Proof Choose max tree $T \subset K$

Let G dual graph: vertices $\longleftrightarrow$ faces of K

edges $\longleftrightarrow$ edges of K not in T.



$$\chi(K) = V_K - E_K + F_K$$

$$= V_T - (E_T + E_G) + V_G = \chi(T) + \chi(G)$$

Exercise: T tree $\Rightarrow \chi(T) = 1$.

G graph $\Rightarrow \chi(G) \leq 1$.

$$\Rightarrow \chi(K) = \chi(T) + \chi(G) \leq 2. \qquad \square$$

Exercise: $L = L_1 \cup L_2$

$$\Rightarrow \quad \chi(L) = \chi(L_1) + \chi(L_2) - \chi(L_1 \cap L_2)$$