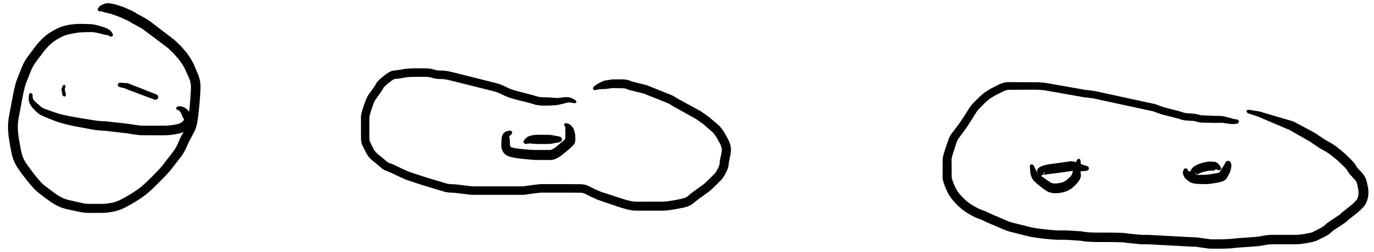
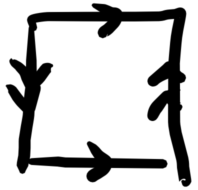
I. Classification of surfaces

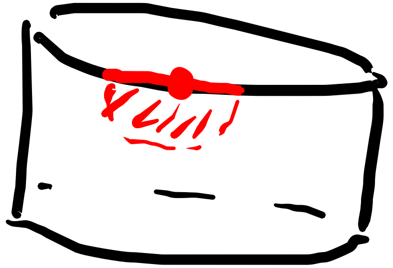
Defn A *closed surface* is a compact, connected, Hausdorff space S s.t. $\forall x \in S$ $\exists\, x \in U \subset S$ s.t. $U \cong \mathbb{R}^2$

Examples:  . . .

$\mathbb{RP}^2$, $K =$ 

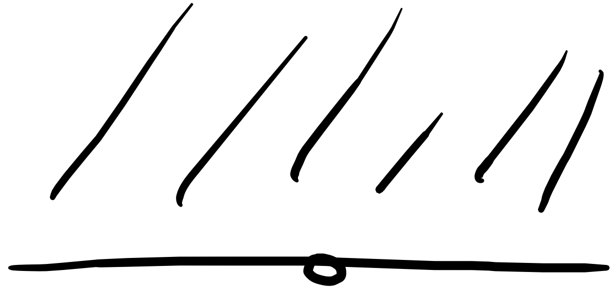
Nonexamples • $\mathbb{R}^2$, $T^2 \setminus pt$ (not compact)

• D^2, M = Möbius, $S^1 \times I$. These are surfaces with boundary



Points p on boundary have neighborhoods $U \cong \mathbb{R} \times [0,\infty) \not\cong \mathbb{R}^2$.

Claim $\mathbb{R}^2 \not\cong \mathbb{R} \times [0,\infty)$.



Pf: Consider $X = \mathbb{R} \times [0,\infty) \setminus \{(0,0)\}$

Check: $\pi_1(X) = \{1\}$. OTOH $\forall p \in \mathbb{R}^2$ $\pi_1(\mathbb{R}^2 \setminus p) \cong \mathbb{Z}$. $\square$

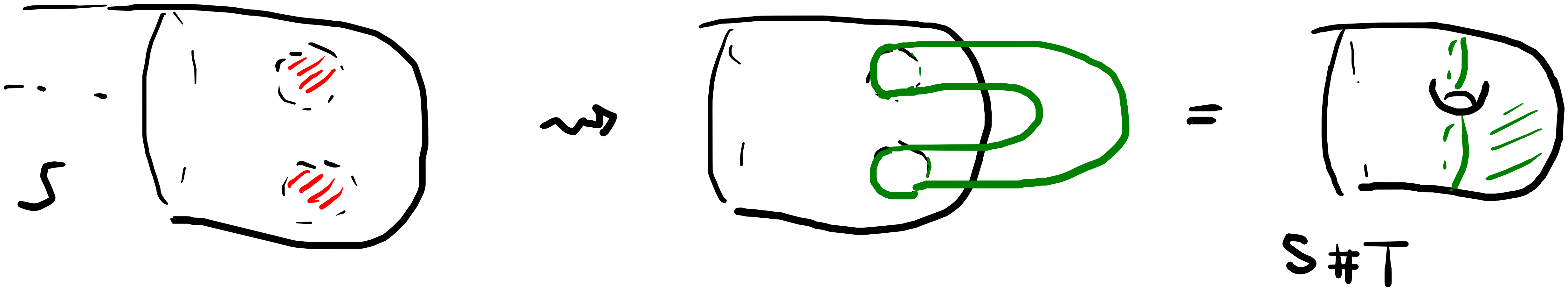
Problem Give a complete list of all closed surfaces. Do same for surfaces with boundary.

Rmk A compact, conn., Hausdorff X locally $\cong \mathbb{R}$ is S^1.

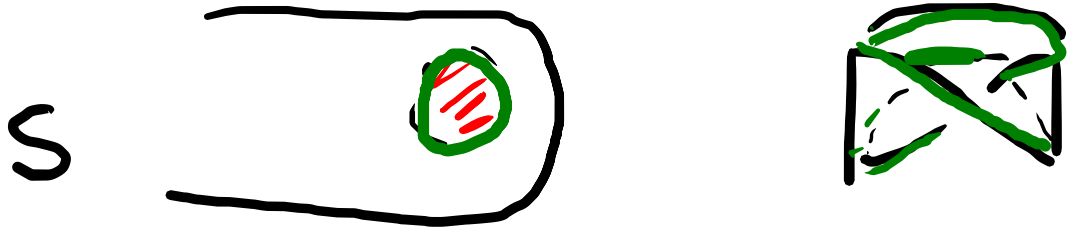
Surface operation

Fix S surface.

(1) Annulus attachment: remove 2 open disk and glue on an annulus A



(2) Möbius attachment: remove disk and glue M

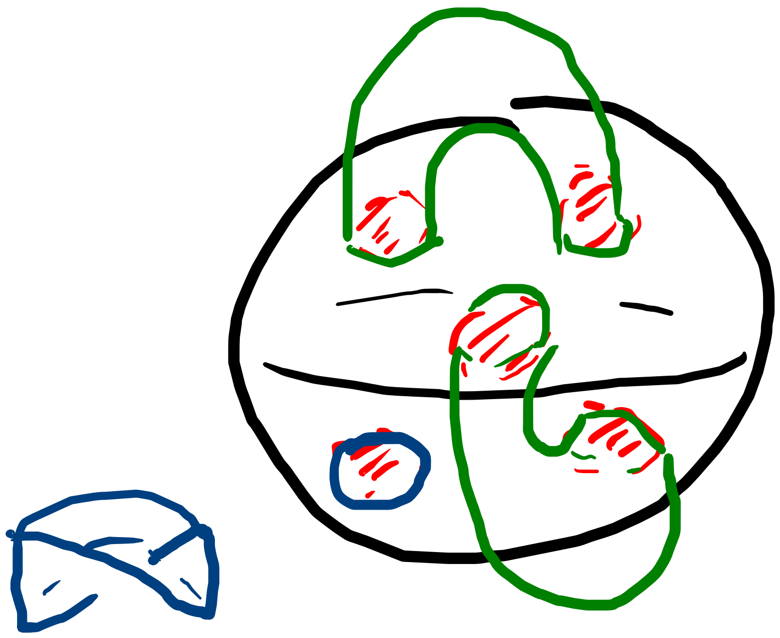


③ Connected sum :

S_1, S_2 surfaces, remove disk from each and glue

Thm (classification of closed surfaces)

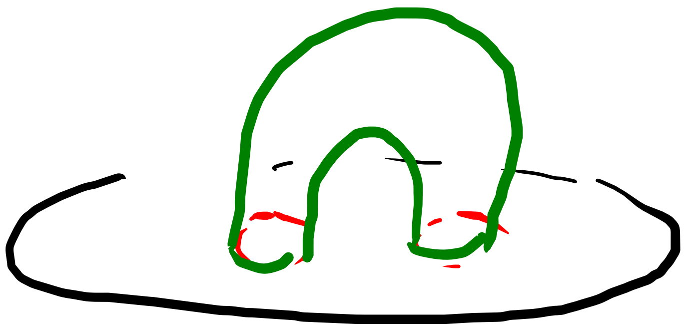
Every closed surface is obtained from S^2 by applying operations (1), (2) finitely many times.



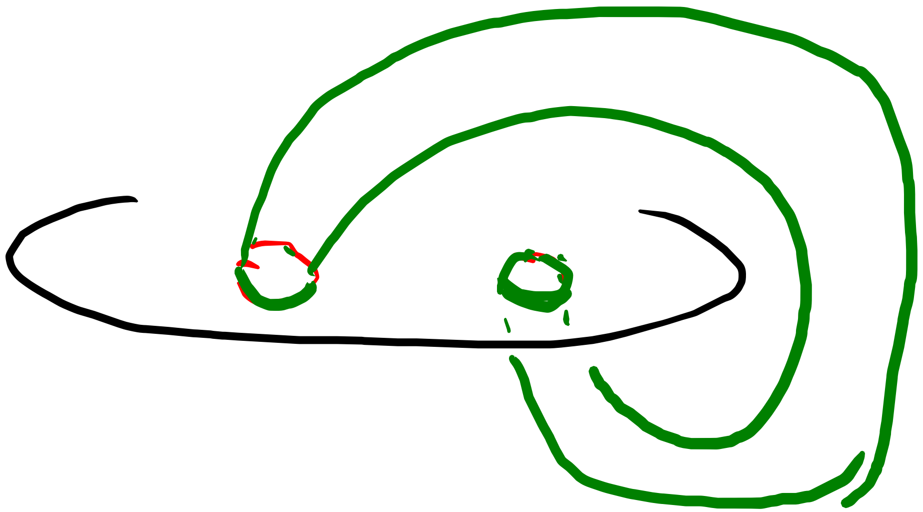
Rmk This list is redundant.

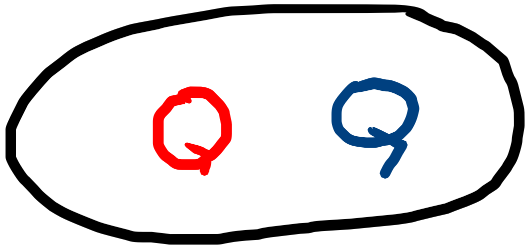
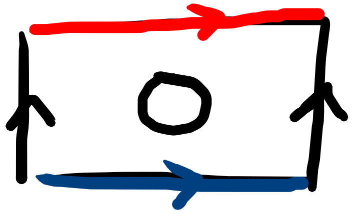
Redundancy / well-definedness

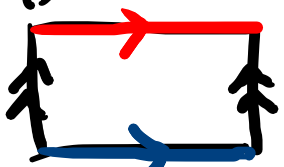
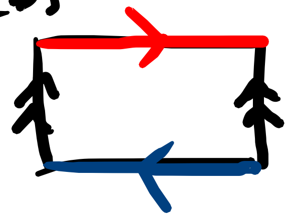
- Annulus attachment is not well-defined:

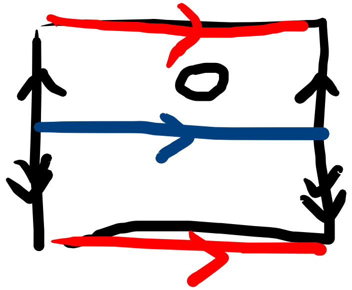


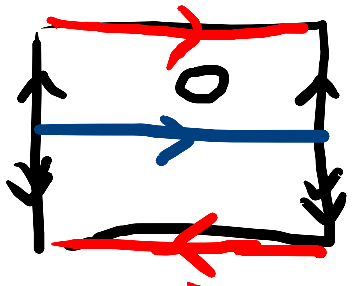
$D^2 \subset S$



 $=$ 

glue (a)  or (b) 

(a)  $= T^2 \setminus D^2$

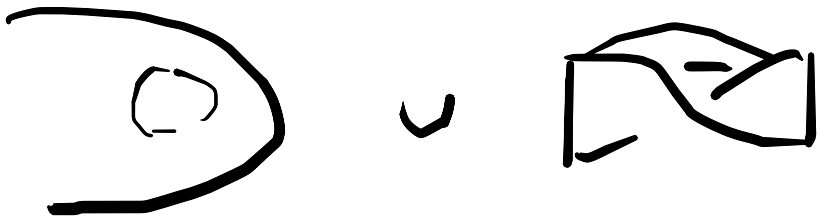
(b) $K \setminus D^2 =$ 

Thus

Annulus attachment (a) is $S \rightsquigarrow S \# T$

(b) is $S \rightsquigarrow S \# K$

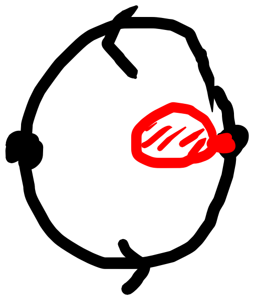
- Möbius attachment as # operation.

S 

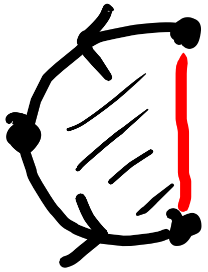
Möbius attachment is $S \rightsquigarrow S \# \mathbb{R}P^2$

Recall $\mathbb{R}P^2 = M \cup D^2$

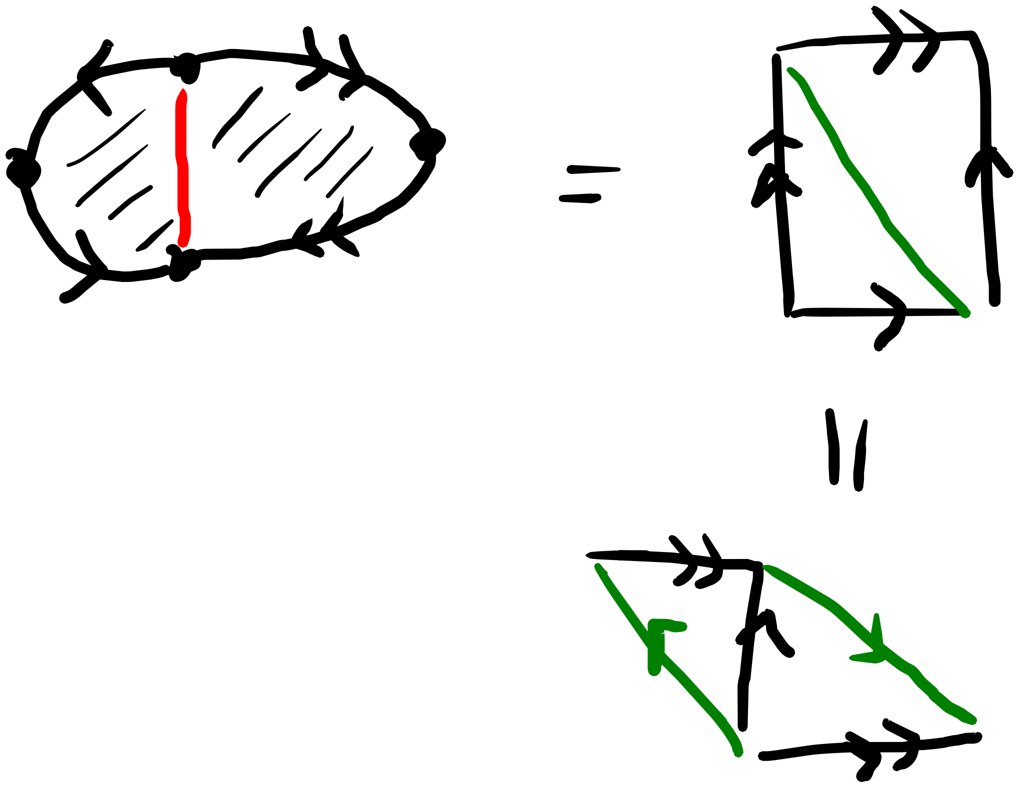
- $\mathbb{RP}^2 \# \mathbb{RP}^2 = K$



$\mathbb{RP}^2$



$\mathbb{RP}^2 \setminus D^2$



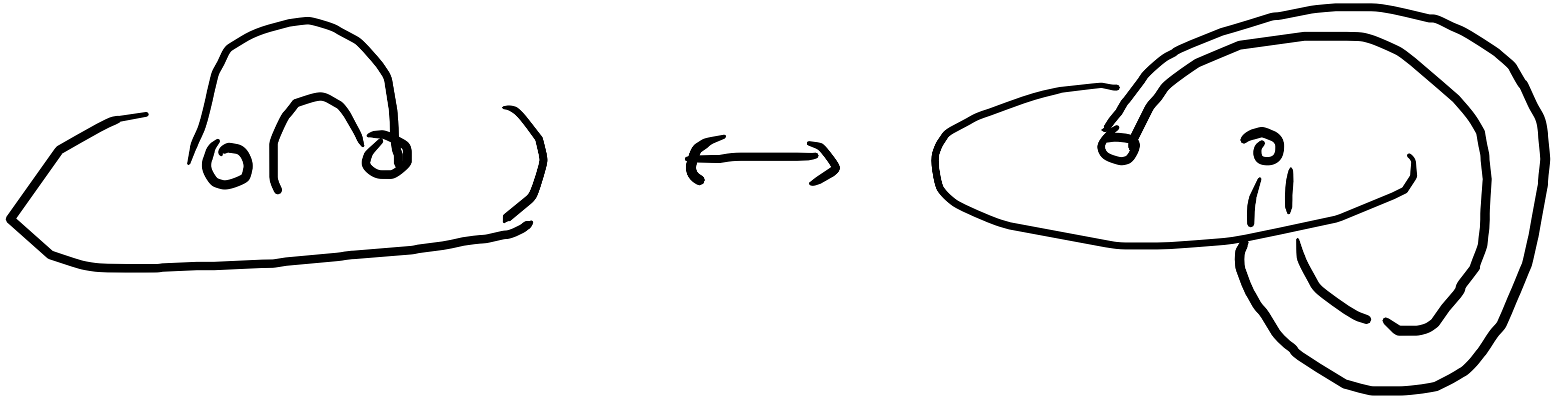
$= K.$

Consequently

$S \longmapsto S \# K$ same as

$S \longmapsto S \# (\mathbb{RP}^2 \# \mathbb{RP}^2)$

- $\mathbb{RP}^2 \# T^2 \cong \mathbb{RP}^2 \# K$



$\mathbb{RP}^2$ is 1-sided ("non-orientable")

$\|$

$M \cup D^2$

__Cor__ For $n \geq 1$.

$$(T^2)^{\# m} \# (\mathbb{RP}^2)^n \cong (K)^{\# m} \# (\mathbb{RP}^2)^{\# n}$$

$$\cong (\mathbb{RP}^2)^{\# 2m+n}.$$

__Thm__ (improved classification of surfaces) Every closed surface is either S^2, $(T^2)^{\# m}$, or $(\mathbb{RP}^2)^{\# k}$ __and__ no two of these are $\cong$.

<u>Rmk</u> $S^2 \# T^2 \cong T^2$

In general $S^2 \# S \cong S$.