I. Barycentric subdivision

Today: $\pi_1(|K|, p) \cong E(K, p)$

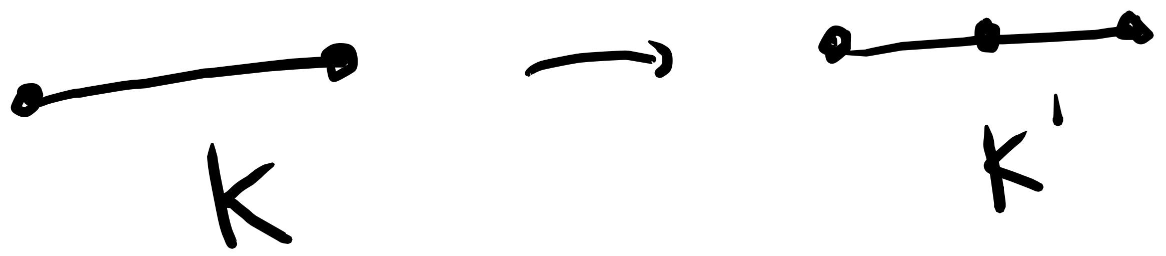
<u>Subdivision</u>.

input: simplicial complex K

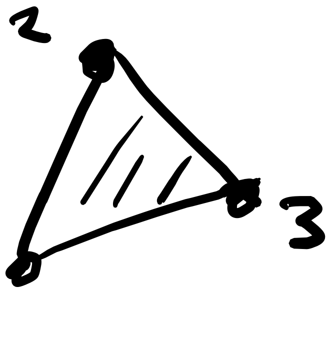
output: complex K' with $|K'| \cong |K|$ and

$|K'|$ has "smaller" simplices

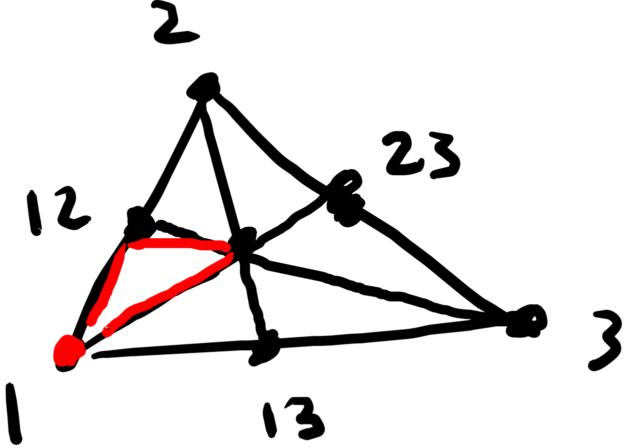
<u>Ex</u>.



$K =$



$K' =$



Formalize: here K has simplices 1, 2, 3

12, 13, 23

123

For K a simplex,

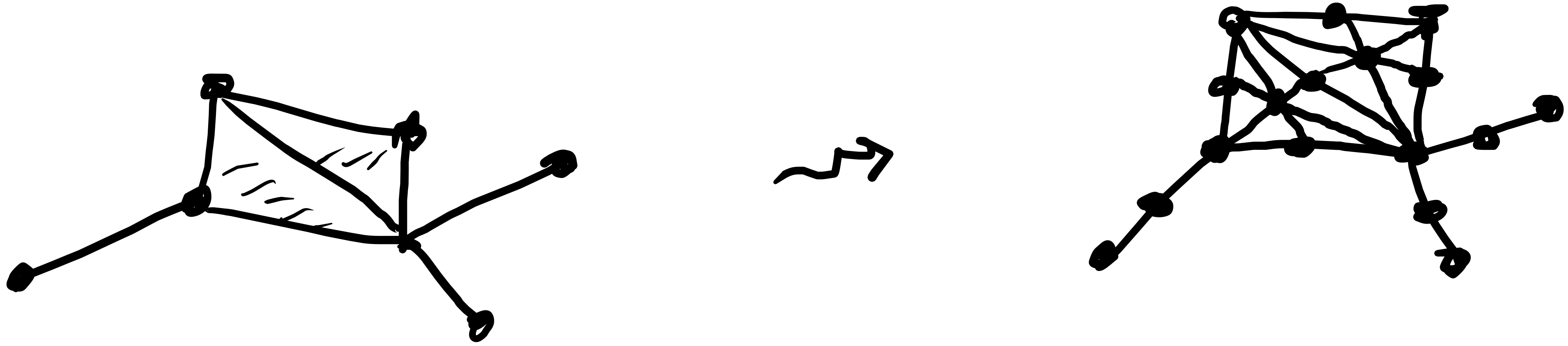
K' has vertices v_σ for $\sigma \in K$ simplex,

If $\sigma_0, \dots, \sigma_n \in K$ simplices and $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$

then $\{v_{\sigma_0}, \dots, v_{\sigma_n}\} \in K'$. eg above $\{v_1, v_{12}\} \in K'$

$\{v_1, v_{12}, v_{123}\} \in K'$, $\{v_1, v_{23}\} \notin K'$

For general K, do this procedure on each simplex



This procedure is barycentric subdivision

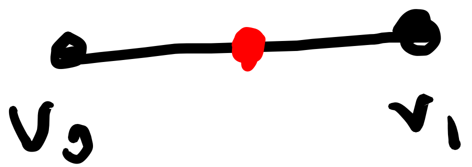
Fix K, $|K| \subset \mathbb{R}^N$

For $\sigma \in K$ w/ vertices $v_0, \dots, v_n$

$|\sigma|$ = simplex spanned by $v_0, \dots, v_n \in \mathbb{R}^N$

Barycenter of $|\sigma|$ is $v_\sigma = \frac{1}{n+1}(v_0 + \dots + v_n)$.

eg $\frac{1}{2}(v_0 + v_1) = v_0 + \frac{1}{2}(v_1 - v_0)$



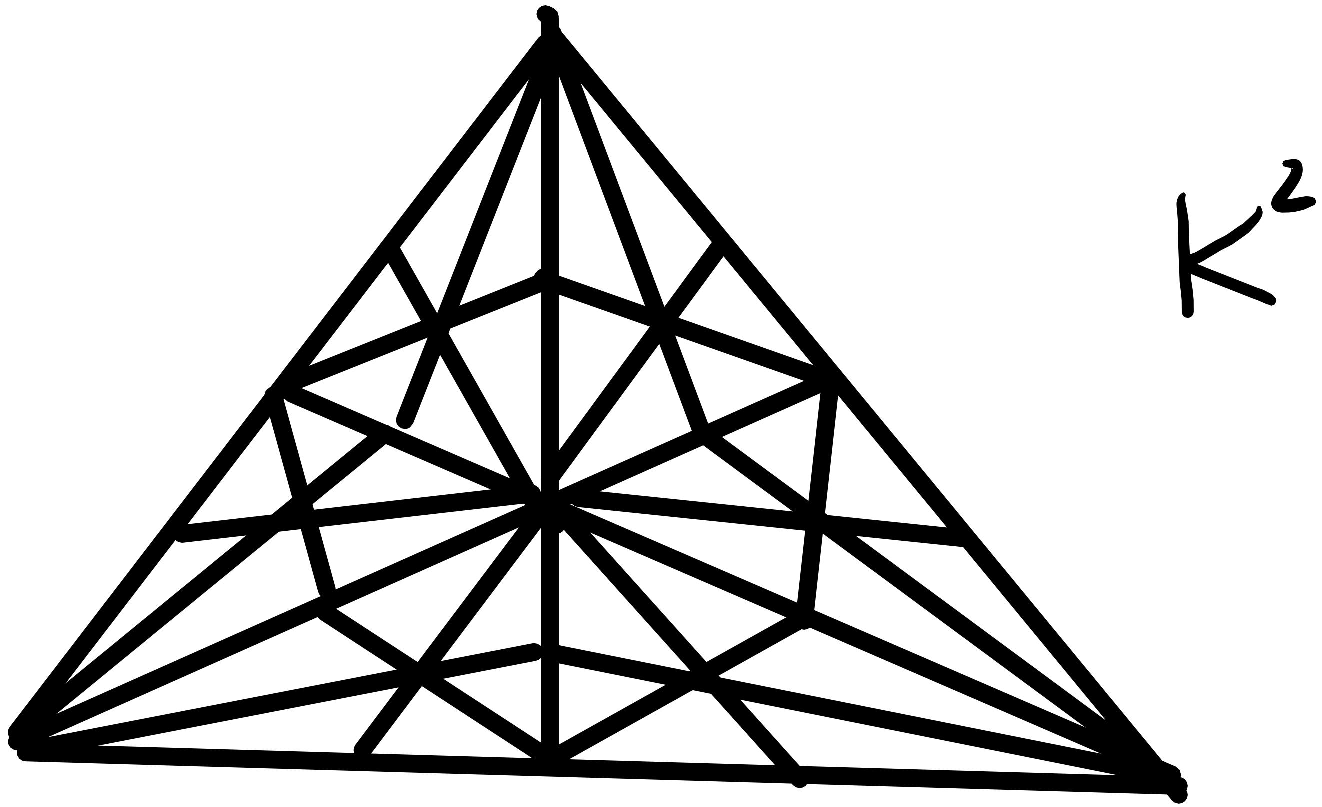
Given $|K| \subset \mathbb{R}^N$ $(v_0, \dots, v_n \in \mathbb{R}^N)$

can choose for $v_\sigma \in K'$ vertex

$$v_\sigma = \frac{1}{n+1}(v_0 + \dots + v_n) \in \mathbb{R}^N$$

Find $|K'| = |K|$ as subsets of $\mathbb{R}^N$.

Note that subdivision can be iterate



write K^r for the rth iterate of subdivision

Thm $f: |L| \longrightarrow |K|$ any map then

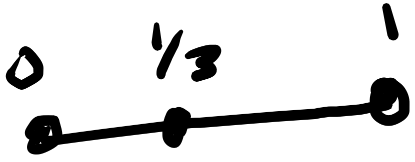
$\exists$ $r \gg 0$ st $f: |L^r| = |L| \longrightarrow |K|$

has a simplicial approximation. $s: |L^r| \longrightarrow |K|$

$s(x) \in \mathrm{carr}(f(x))$.

Ex $\underset{|L|}{f: [0,1]} \longrightarrow \underset{|K|}{[0,1]}$

$x \longmapsto x^2$



L

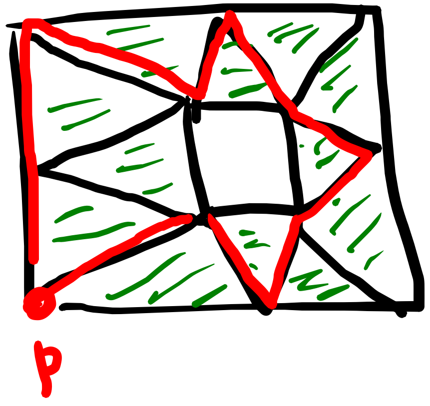
K

f has no simplicial approx but

$|L^2| \longrightarrow |K|$ does (exercise)

II Edge group theorem

Thm K simplicial complex $\pi_1(|K|, p) \cong E(K, p)$.



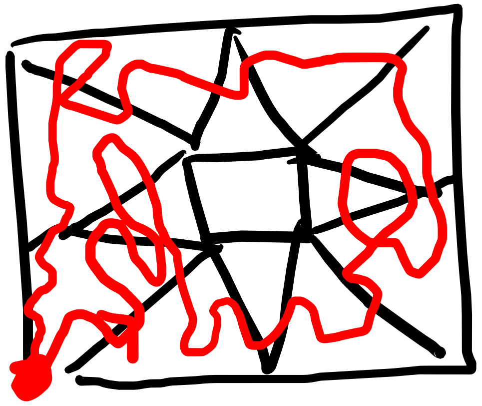
Proof $\Phi : E(K,p) \longrightarrow \pi_1(|K|, p)$

edge loop $\longmapsto$ $[f : [0,1] \longrightarrow |K|]$

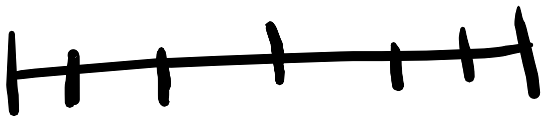
$p u_1 \cdots u_n p$ piecewise linear traverses edge loop.

Φ is a homomorphism

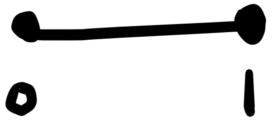
Φ is surjective: Fix $[f] \in \pi_1(|K|, p)$.



WTS: can homotope f to a simplicial map.



$\xrightarrow{s}$ $|K|$.

Set $L =$ 

$|L| = [0,1]$

simplicial approx: $\exists$ $r > 0$ s.t. $f : |L^{r}| \longrightarrow |K|$.

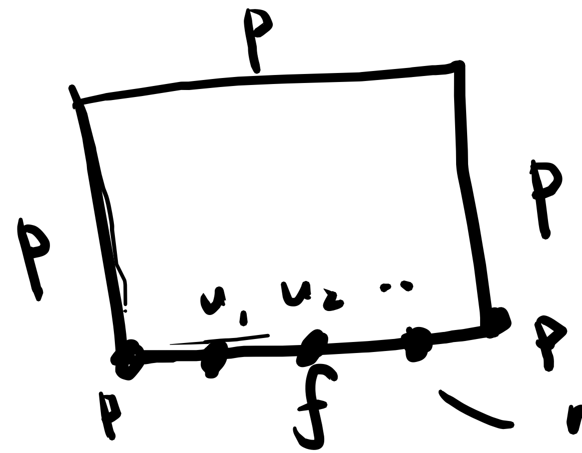
has a simplicial approx s. Know $s \sim f$

Note $s(0) = s(1) = p$

Φ injective

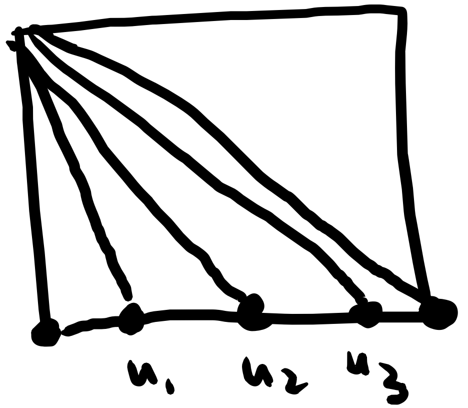
Fix $\varepsilon = p u_1 \cdots u_n p$

Suppose $\Phi(\varepsilon) = [f] = [p]$

$\Rightarrow \exists$ homotopy 

$\xrightarrow{h} |K|.$

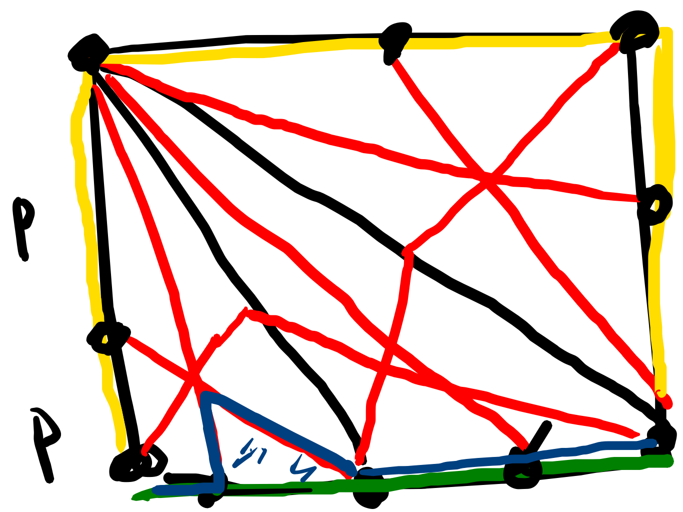
map to vertices

Consider $L =$ 

$h : |L| \longrightarrow |K|.$

Simplicial approx: $h : |L^r| \longrightarrow |K|$ has simplicial approx s.

ie up to homotopy, $h : |L^r| \longrightarrow |K|$ is simplicial.



green / yellow edge paths are equivalent in L^r

Since h is simplicial then $h(\text{green})$ & $h(\text{yellow})$ are equivalent edge loops.

but $h(\text{green}) = f$ and $h(\text{yellow})$ $ppp\cdots \sim p$

$\square$