I. Simplicial complexes & triangulations

Done:

X	$\pi_1(X)$
$\mathbb{R}^n$	0
$S^n, n \geq 2$	0
S^1	$\mathbb{Z}$
T^2	$\mathbb{Z} \times \mathbb{Z}$

Todo:

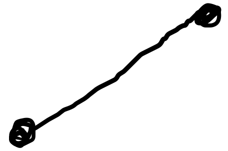
X	$\pi_1(X)$
$\mathbb{RP}^n$	$\mathbb{Z}/2\mathbb{Z}$
Klein bottle	$\langle a, b \mid aba^{-1}b = 1 \rangle$
$\infty = S^1 \vee S^1$	$\mathbb{Z} * \mathbb{Z}$
$\mathbb{R}^3 \setminus$ 	??

<u>Next</u>: develop systematic way to compute $\pi_1(X)$.

New tools: triangulations, simplicial approximation, van Kampen theorem

Defn The **k-Simplex** spanned by $v_0, \dots, v_k \in \mathbb{R}^n$ is

$$\left\{ t_0 v_0 + \dots + t_k v_k \;\middle|\; \begin{array}{l} 0 \le t_i \le 1, \\ t_0 + \dots + t_k = 1 \end{array} \right\}$$

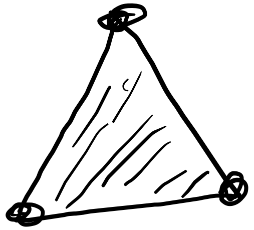


v_1

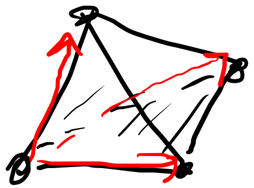
v_0

$$\{ t_0 v_0 + t_1 v_1 \mid t_0 + t_1 = 1 \} = \{ s v_0 + (1-s) v_1 \mid 0 \le s \le 1 \}.$$

1-simplex



2 Simplex



3 simplex

v_0

0-simplex

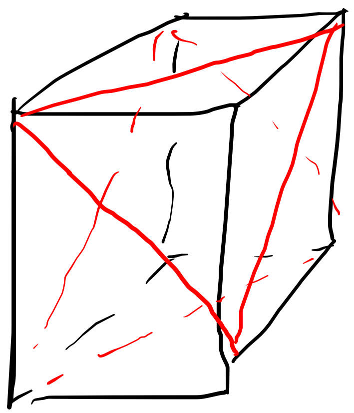
only consider case when $v_0, \dots, v_k$ is in general position, ie

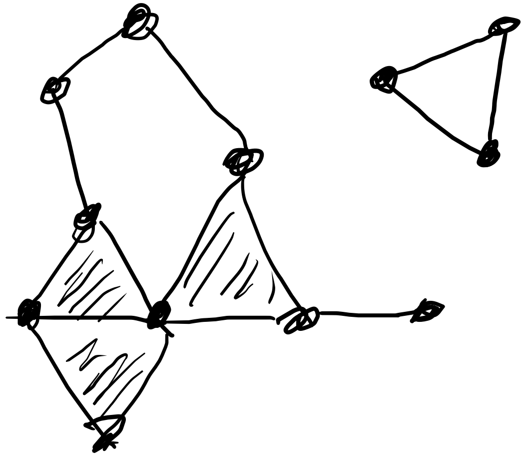
$v_0, \dots, v_k$ not contained in $(k-1)$-dimensional plane, ie

$v_1 - v_0, \dots, v_k - v_0$ are linearly independent.

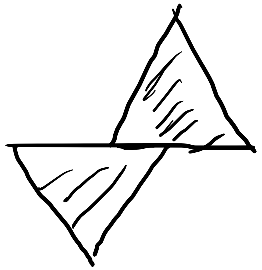
<u>Informal definition</u> A <u>simplicial complex</u> K is a topological space obtained from gluing simplicies "nicely"

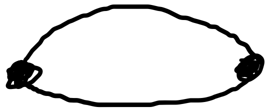
ex:





nonexamples:





want simplex in K determined by its vertices.

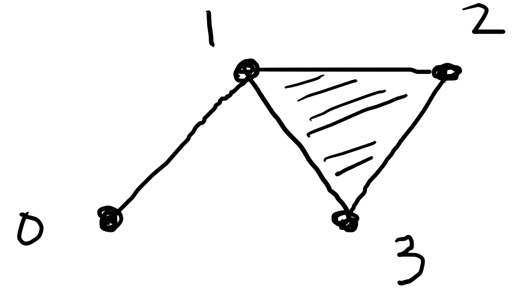
Formal definition V finite set (vertices)

A **simplicial complex** K with vertex set V is a collection of nonempty subsets of V that includes $\{v\}$ for $v \in V$ and if $X \in K$ and $Y \subset X$ then $Y \in K$. $\left(K \subset P(V) \setminus \phi \right)$

Rmk. K is not a space (yet)

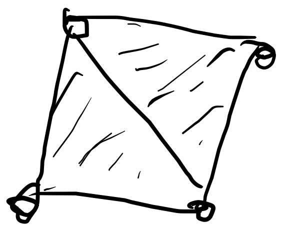
Eg. $V = \{0,1,2,3\}$

$$K = \left\{ \begin{array}{l} \{0\}, \{1\}, \{2\}, \{3\}, \\ \{0,1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \end{array} \right\}$$



if we remove $\{0,1\}$ from K then still get simplicial complex, but not if we remove $\{1,3\}$.

Ex. $V = \{0,1,2,3\}$ $\quad L = P(V) \setminus \emptyset$.



Defn / construction (geometric realization)

K simplicial complex with $V = \{v_0, \dots, v_n\}$

Choose $V \hookrightarrow \mathbb{R}^n$ general position. For each $X = \{v_{i_0}, \dots, v_{i_k}\} \in K$

let $|X|$ = simplex spanned by X. The geometric realization of

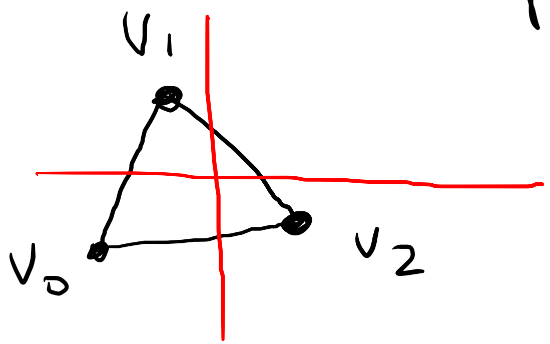
K is

$$|K| := \bigcup_{X \in K} |X| \subset \mathbb{R}^n$$

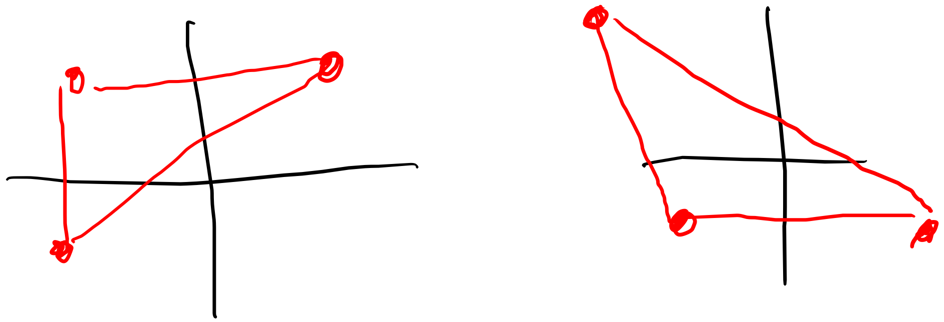
topologized as a subspace

Ex. $V = \{v_0, v_1, v_2\}$ $\quad M = \{\{v_i\}, \{v_i, v_j\}\}$

$|M| \cong S^1$

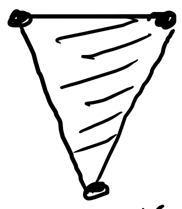


Rmk Choosing different $V \hookrightarrow \mathbb{R}^n$ results in a topologically equivalent space.

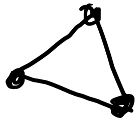
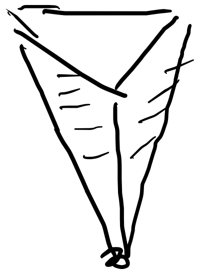


Ex K simplicial complex $\overset{\text{vertices } V}{V}$. Define **Cone** $C(K)$ simplicial complex w/ vertices $V' = V \cup \{*\}$ and

$$C(K) = K \cup \{\text{all subsets containing } *\}.$$

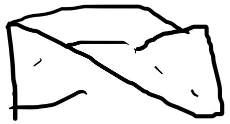
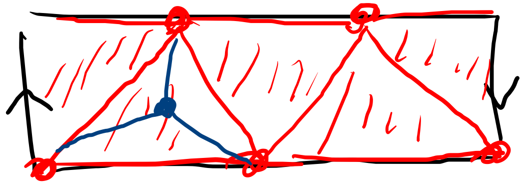
eg $K =$  $\mathrm{Cone}(K) =$ 

$*$

$K =$  $C(K) =$ 

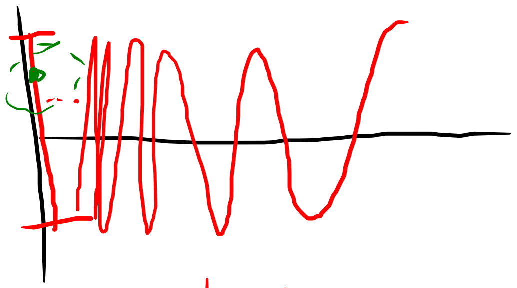
Triangulations

A <u>triangulation</u> of a space X is a simplicial complex K with a top. equiv. $|K| \cong X$.

<u>Ex</u>.  

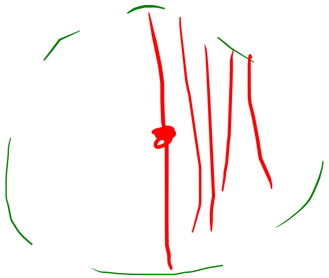
note: a given X may have many triangulations

<u>Nonexample</u>: a given X may not have any triangulation

eg 

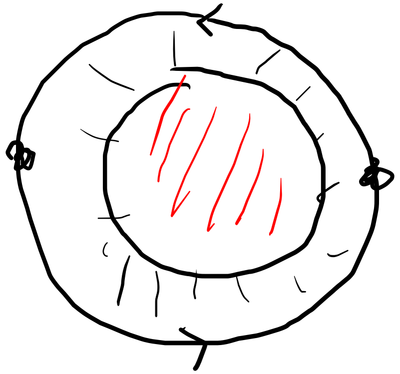
topologist sine curve

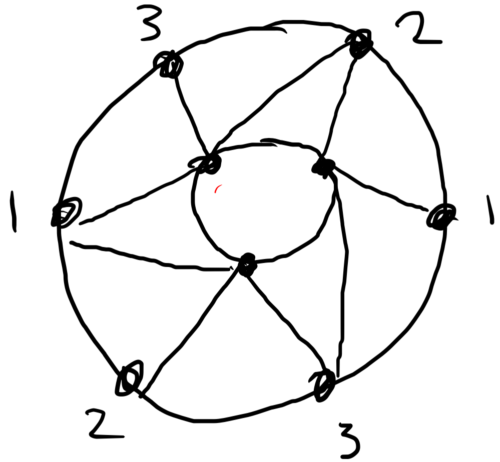
has no triangulation.



Ex $\mathbb{R}P^2 \cong D^2 \cup_\partial \text{Mobius}.$

$$= \text{Mobius} \cup C(\partial\,\text{Mobius})$$





II Fundamental group of a triangulated space

Fix space X, simplicial complex K, $|K| \cong X$

π_1 is a top. invar $\Longrightarrow$ $\pi_1(X) \cong \pi_1(|K|)$.

Want: compute $\pi_1(|K|)$ in a combinatorial way.

An <u>edge path</u> in K is a sequence of vertices $u_1 \cdots u_d$ s.t.

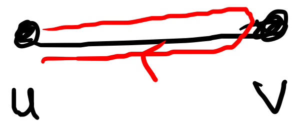
~~for~~ each $i = 1, \ldots, d-1$ either $u_i = u_{i+1}$ or $\{u_i, u_{i+1}\} \in K$.

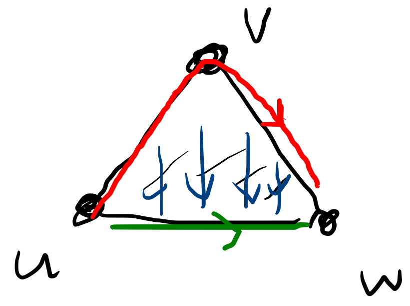
an edge path of form $p u_1 \cdots u_d p$ is called an <u>edge loop</u>

Say two edge paths are <u>equivalent</u> if they differ by the following moves:

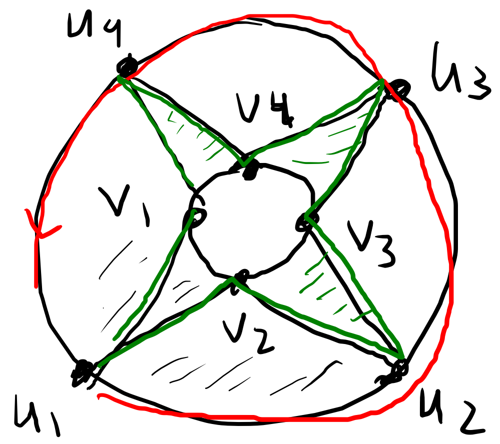
$uu \longleftrightarrow u$ $\qquad$ $uvu \longleftrightarrow u$ $\qquad$ $uvw \longleftrightarrow uw$

if $\{u,v,w\} \in K$.





Ex. $|K| \cong S^1 \times [0,1]$



$u_1 u_2 u_3 u_4 u_1 \sim$

$u_1 v_2 u_2 v_3 u_3 v_4 u_4 v_1 u_1$

$\sim \; u_1 v_2 v_3 v_4 v_1 u_1$

The <u>edge group</u> of simplicial complex K with base vertex p is

$$E(K,p) = \{\text{edge loops at } p\}/\sim$$

(equivalence through loops at p).

group operation: concatenation

$$[pu_1 \cdots u_d p] \cdot [pv_1 \cdots v_n p] = [pu_1 \cdots u_d p v_1 \cdots v_n p]$$

identity: $[p]$, inverses: $[pu_1 \cdots u_d p]^{-1} = [pu_d \cdots u_1 p]$

$$pu_1u_2p\,pu_2u_1p \sim pu_1\underbrace{u_2p}\,u_2u_1p \sim pu_1\underbrace{u_2}u_1p \sim pu_1p \sim p.$$

__Thm__ (edge group) $E(K,p) \cong \pi_1(|K|,p)$.

combinatorial!