I. Path lifting lemma

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

<u>Last time</u> $\quad \pi_1(S^1, 1) \cong \mathbb{Z}$

- Step 1 (path lifting) Given $f : [0,1] \longrightarrow S^1$ loop based at $\underline{1}$

  $\exists!\ \tilde{f} : [0,1] \longrightarrow \mathbb{R}$ s.t. $\tilde{f}(0) = 0$ and $p \circ \tilde{f} = f$

- Step 2 : define $\Phi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ by $\Phi([f]) = \tilde{f}(1)$.

  homomorphism

- Step 3 : Φ bijective.

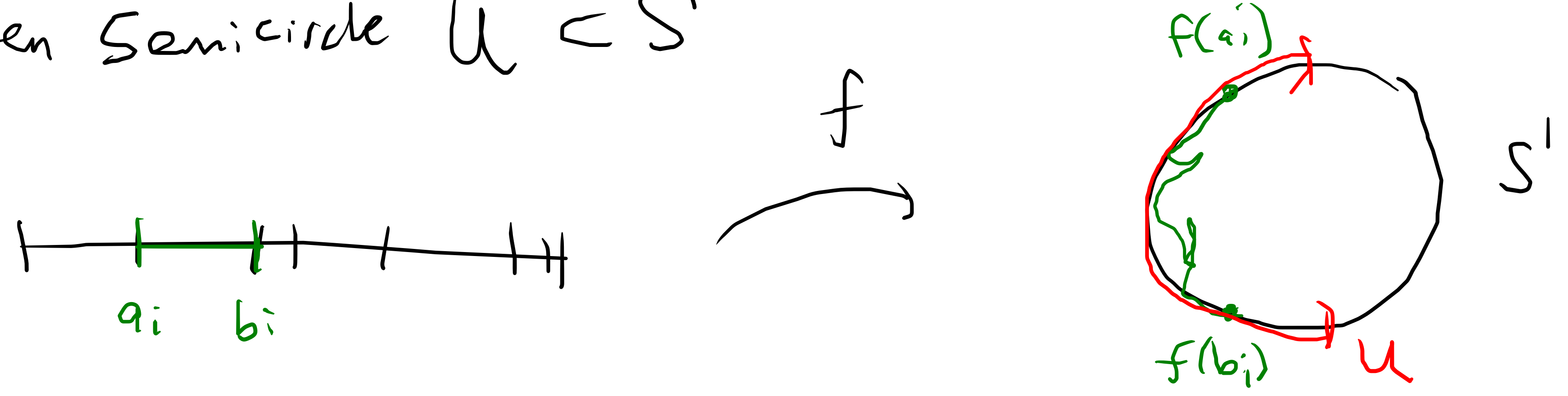
Prop (path lifting) Given $f:[0,1] \to S^1$ loop based at 1

$\exists!\ \tilde{f}:[0,1] \to \mathbb{R}$ s.t. $\tilde{f}(0) = 0$ and $p \circ \tilde{f} = f$

Proof Fix $f:[0,1] \to S^1$

- from last week $\exists$ partition $[0,1] = \bigcup_{i=1}^{k} [a_i, b_i]$ s.t.

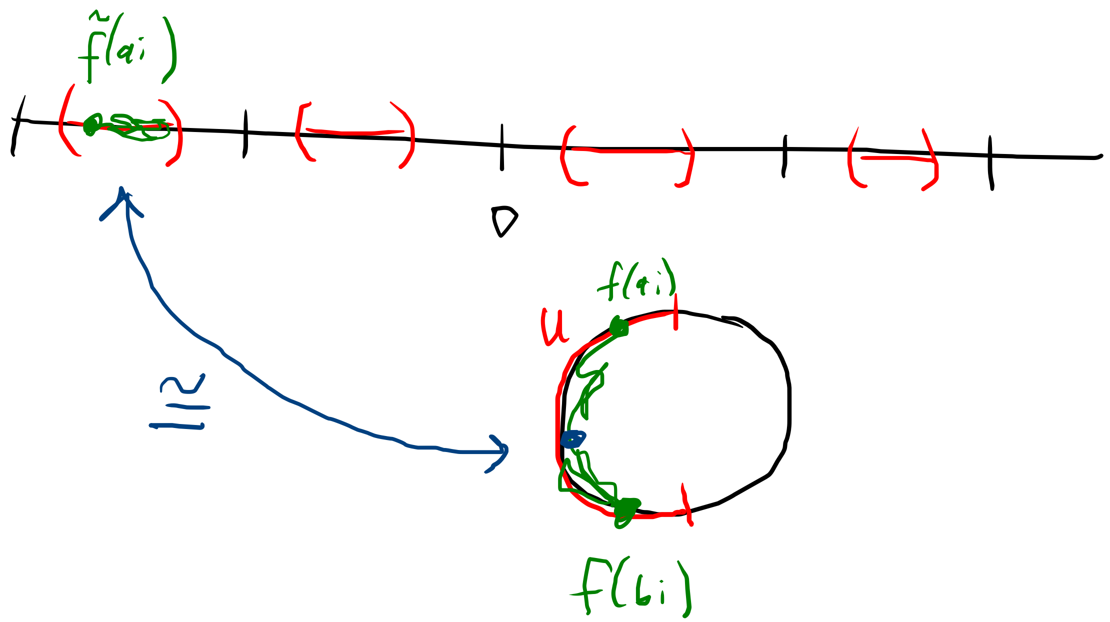
$f([a_i, b_i]) \subset$ open semicircle $U \subset S^1$



- observe that for $p: \mathbb{R} \to S^1$

$$U = \{ e^{2i\pi t} \mid \theta_0 < t < \theta_0 + \pi \}$$

$$p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} \left(\theta_0 + n, \theta_0 + \tfrac{1}{2} + n \right)$$



- define $\tilde{f}$ inductively

suppose $\tilde{f}(a_i)$ is defined. Then define $\tilde{f}$ on $[a_i, b_i]$:

$\cap$

$(\theta_0 + n, \theta_0 + \tfrac{1}{2} + n) \xrightarrow[\cong]{p|} U$. write $r : U \to (\theta_0 + n, \theta_0 + \tfrac{1}{2} + n)$ inverse of $p|$

Define $\tilde{f}(t) = r \circ f(t)$ for $t \in [a_i, b_i]$

$p \circ \tilde{f}(t) = p \circ r \circ f(t) = f(t)$ so $\tilde{f}$ is a lift of f on $[a_i, b_i]$.

<u>Note</u> $\tilde{f}(0) = 0$ starts the induction.

- uniqueness: we didn't make any choices. $\square$

<u>Rmk</u>. The property of $\mathbb{R} \xrightarrow{p} S^1$ used above movitates:

<u>Defn</u> A map $p: X \to Y$ is a <u>covering map</u> if $\forall y \in Y$ $\exists$ open $\mathcal{U} \ni y$ so that $p^{-1}(\mathcal{U}) \cong \mathcal{U} \times \Delta$ where Δ is discrete.

(eg $S^2 \longrightarrow \mathbb{R}P^2$ is a covering map)

Cor $\pi_1(T^2) \cong \mathbb{Z}^2$

Prop $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

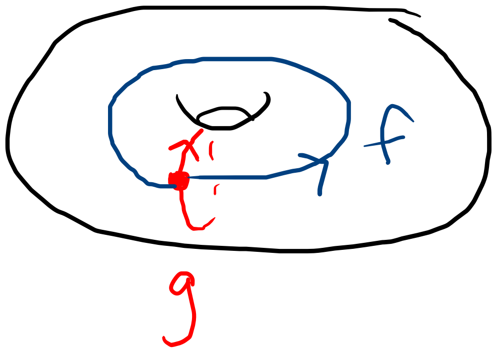
$$\left(\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z} \right)$$

About Prop : $f : [0,1] \longrightarrow X \times Y$ loop based at $(p,q) \in X \times Y$

$f = (f_1, f_2)$ Define $\pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$

$$[(f_1, f_2)] \longmapsto ([f_1], [f_2])$$ $\square$

For T^2



$$\left.\begin{aligned} f(t) &= (e^{2\pi i t}, 1) \\ g(t) &= (1, e^{2\pi i t}) \end{aligned}\right\} \text{ generate } \pi_1(T^2)$$

Cor $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n > 2$.

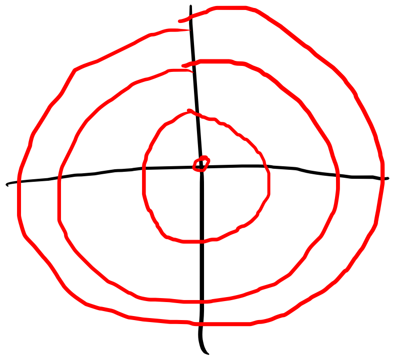
(already proved $\mathbb{R}^2 \not\cong \mathbb{R}^1$)

Proof For contradiction suppose $\mathbb{R}^2 \cong \mathbb{R}^n$ $n \geq 3$.

Then $\mathbb{R}^2 \setminus 0 \cong \mathbb{R}^n \setminus 0 \cong S^{n-1} \times (0,\infty)$

$\quad\ \ \cong$

$S^1 \times (0,\infty)$



$$\Longrightarrow \quad \pi_1(S^1 \times (0,\infty)) \cong \pi_1(S^{n-1} \times (0,\infty)) \cong \pi_1(S^{n-1}) = 0$$

$$\cong$$

$$\mathbb{Z} = \pi_1(S^1) \qquad \text{\#}$$

$\square$

Rmk This argument can't show $\mathbb{R}^3 \ncong \mathbb{R}^4$
(homology)

Cor (D^2 retraction) There is no (continuous) map

$r: D^2 \longrightarrow S^1$ s.t. $r|_{S^1} = id_{S^1}$

(such a map is called a <u>retract</u>)

$$\begin{tikzcd} & S^1 \\ S^1 \arrow[ur, "id"] \arrow[r, hook, "j"'] & D^2 \arrow[u, dashed, "r?"'] \end{tikzcd}$$

Proof Suppose r exists.

Then $r \circ j = id_{S^1} \implies$

$$r_* \circ j_* = (r \circ j)_* = (id_{S^1})_* = id_{\pi_1(S^1)}$$

$$
\begin{tikzcd}
\mathbb{Z} = \pi_1(S^1) \arrow[r, "j_*"] \arrow[dr, "id"'] & \pi_1(D^2) = 0 \arrow[d, "r_*"] \\
 & \pi_1(S^1) = \mathbb{Z}
\end{tikzcd}
$$

Since $\pi_1(D^2) = 0$

$j_* \circ r_*$ is 0.

but $j_* \circ r_* = id_{\mathbb{Z}}$ $\lightning$.

$\square$

<u>Cor to Cor</u> (Brouwer fixed pt)

Any map $f : D^2 \to D^2$ has a fixed point, ie.

$\exists\, x \in D^2$ $f(x) = x$.

<u>Proof</u> Suppose $\exists\, f$ w/ no fixed pts. Define $r : D^2 \to S^1$

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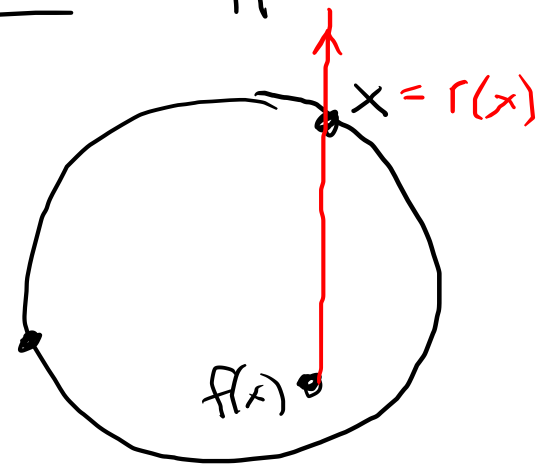
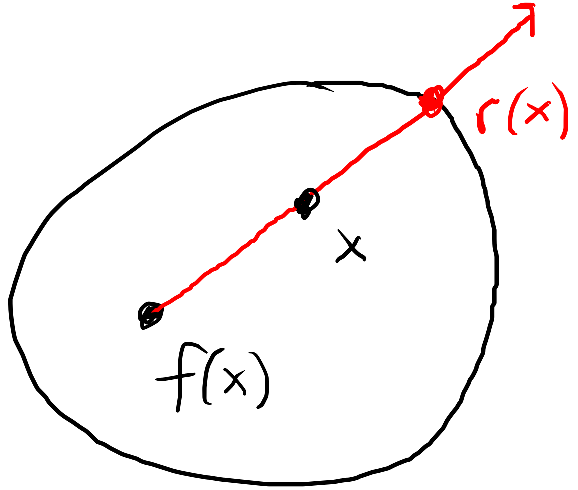


$r(x) =$ intersection with S^1 of ray from $f(x)$ to x

r is a retract $\times$.



$\square$