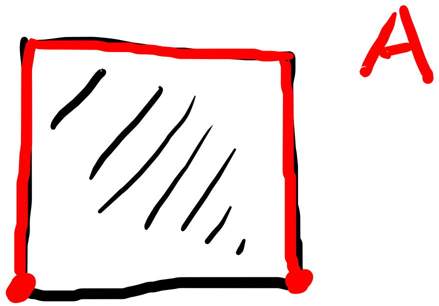
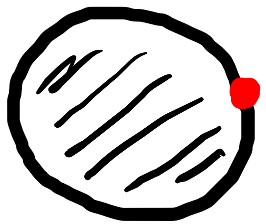
I. More homotopy



Warmup: $X = [0,1] \times [0,1]$

partition $\mathcal{P}$: A, $\{x\}$ for $x \notin A$.

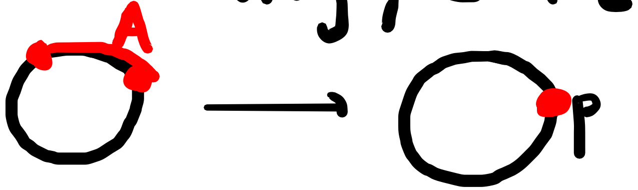
write X/A for $\mathcal{P}$ w/ quotient topology.



Claim $X/A \cong \mathbb{D}^2$

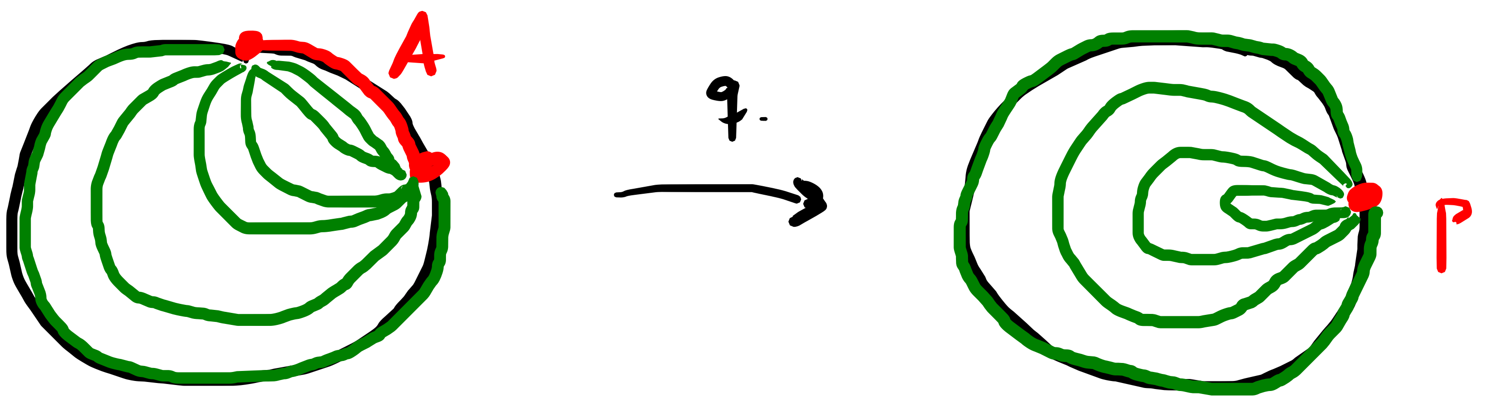
Proof sketch Want $\overset{\text{surj.}}{\text{map}}$ $X \longrightarrow \mathbb{D}^2$ whose partition is $\mathcal{P}$.

Equivalently, since $[0,1]^2 \cong \mathbb{D}^2$, suffices to find surj $\mathbb{D}^2 \xrightarrow{q} \mathbb{D}^2$



s.t. $q(A) = p$ and $q|_{A^c} : A^c \longrightarrow \mathbb{D}^2 \setminus p$ bij.

Defining $q : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$



<u>Recall</u> • fundamental group of X based at $p \in X$

$$\pi_1(X, p) = \{ f : [0,1] \longrightarrow X \mid f(0) = p = f(1) \} / \text{homotopy}$$

• $[c] \in \pi_1(X, p)$ homotopy class of constant is identity elmt.

if $f \sim c$ say f is nullhomotopic.

Observations

(1)

$$\{ f : [0,1] \to X \mid f(0) = p = f(1) \} \overset{1-1}{\longleftrightarrow} \{ g : \overset{\mathbb{C}}{\overset{\cup}{S^1}} \to X \mid g(1) = p \}$$

Given f, define $\hat{f} : S^1 \to X$ by $\hat{f}(e^{2\pi i t}) = f(t)$ for $t \in [0,1]$

$\hat{f}$ well defined b/c $f(0) = f(1)$.

$\hat{f}$ continuous b/c $f = \hat{f} \circ q$ is continuous.

$$\begin{tikzcd}
{[0,1]} \arrow[d, "q"'] \arrow[dr, "f"] & \\
S^1 \arrow[r, "\hat{f}"'] & X
\end{tikzcd}$$

$q(t) = e^{2\pi i t}$ quotient map

$$\begin{tikzcd}
{[0,1]} \arrow[d, "q"'] \arrow[dr, "g \circ q"] & \\
S^1 \arrow[r, "g"'] & X
\end{tikzcd}$$

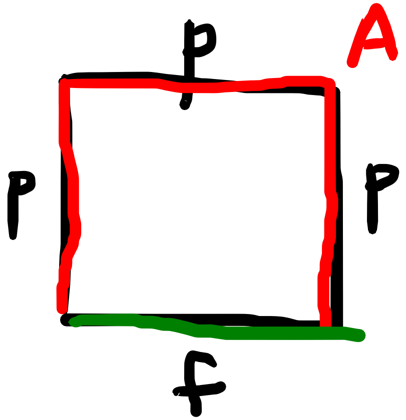
Conversely, $g : S^1 \to X$, define $\hat{g} = g \circ q$

(2) $f : [0,1] \to X$ nullhomotopic loop $\iff$
$$\begin{tikzcd} S^1 \arrow[r, "\hat{f}"] \arrow[d, hook] & X \\ D^2 \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$
extends to D^2

$$\varphi|_{S^1} = \hat{f}.$$

<u>Proof</u> ($\Rightarrow$) Fix f nullhomotopic.

$\exists$ homotopy



$$\begin{tikzcd} Y := [0,1] \times [0,1] \arrow[r, "F"] \arrow[d] & X \\ Y/A \cong D^2 \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

$\varphi(\{x\}) = F(x)$

$\varphi(A) = F(A) = p.$

observe that $\varphi|_{S^1} = \hat{f}$

so $\hat{f}$ extends ✓

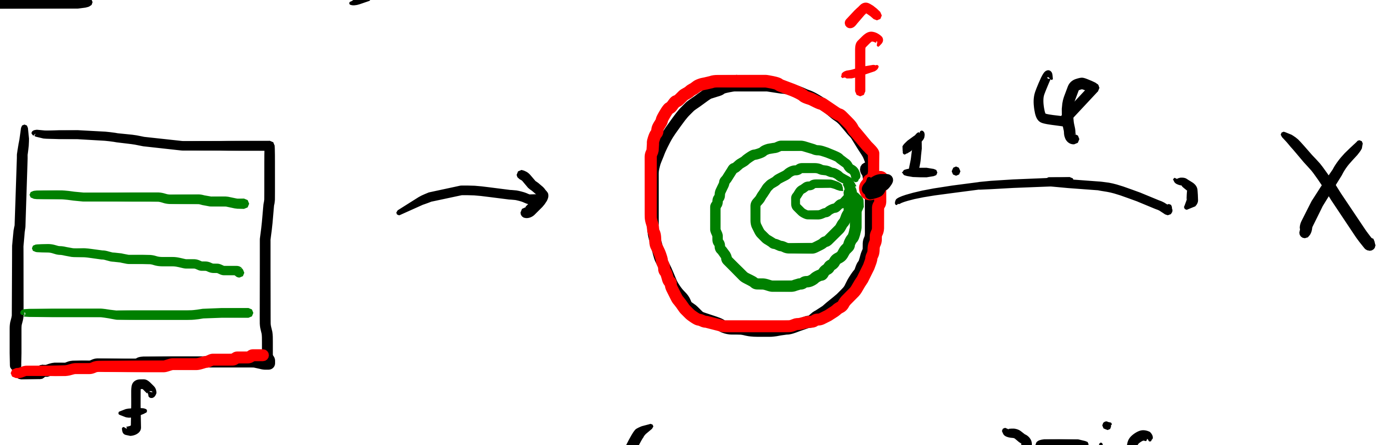
(2) $f : [0,1] \longrightarrow X$ nullhomotopic loop $\iff$

$$\begin{tikzcd} S^1 \arrow[r, "\hat{f}"] \arrow[d, hook] & X \\ \mathbb{D}^2 \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

$\hat{f}$ extends to $\mathbb{D}^2$

$$\varphi|_{S^1} = \hat{f}.$$

<u>Proof</u> $(\Longleftarrow)$ Assume $\hat{f}$ extends.



$$F(s,t) = \varphi\big((1-t)e^{2\pi i s} + t\big)$$

$$F(s,0) = \varphi(e^{2\pi i s}) = \hat{f}(e^{2\pi i s}) = f(s)$$

$$F(s,1) = \varphi(1) = p$$

$$F(0,t) = \varphi(1) = p$$

$$F(1,t) = \varphi(1) = p.$$

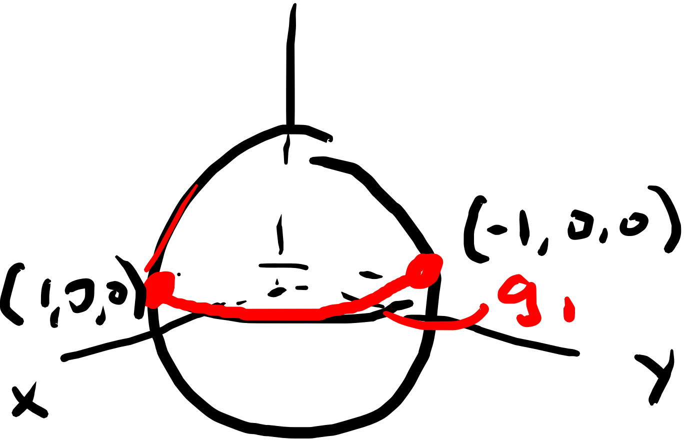
✓

$\square$

Example $\mathbb{R}P^2$ = lines through 0 in $\mathbb{R}^3$

= quotient of S^2 by antipodal map

= D^2 (upper hemisphere) / antipodal pts on ∂.



Consider paths in S^2 $\quad g_1(s) = (\cos \pi s, \sin \pi s, 0)$ $\quad s \in [0,1]$.

$g_2(s) = (\cos 2\pi s, \sin 2\pi s, 0)$

$q : S^2 \longrightarrow \mathbb{R}P^2$ quotient map.

$$\left.\begin{aligned} f_1 &= q \circ g_1 \\ f_2 &= q \circ g_2 \end{aligned}\right\} \text{ loops based at } p = q(1,0,0)$$

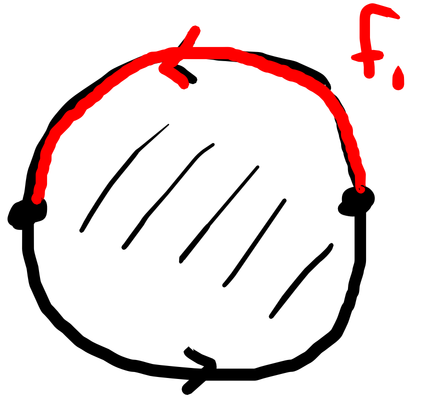
Note $\hat{f}_2$ extends to D^2 b/c $\hat{g}_2$ does:

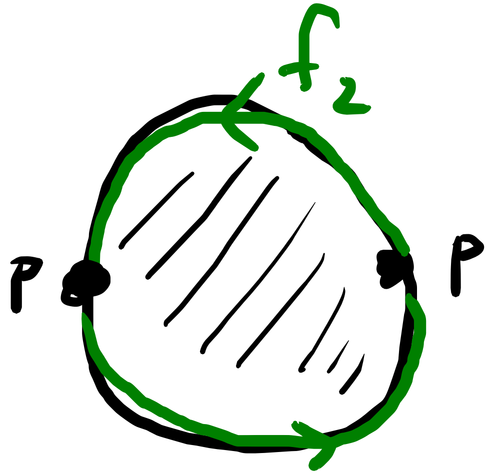
Define $\varphi: \mathbb{D}^2 \longrightarrow S^2$

$(r\cos 2\pi s, r\sin 2\pi s) \longmapsto (r\cos 2\pi s, r\sin 2\pi s, \sqrt{1-r^2})$

note $\varphi|_{\partial D^2} = \hat{g}_2$

Then $q \circ \varphi : D^2 \longrightarrow \mathbb{R}P^2$ $\quad q \circ \varphi|_{\partial \mathbb{D}^2} = q \circ \hat{g}_2 = \hat{f}_2$.





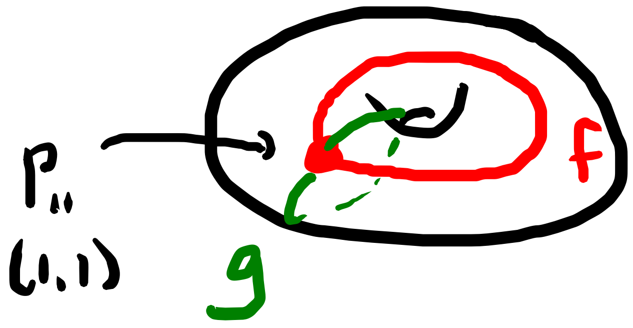
$f_2 = f_1 * f_1$

Thus in $\pi_1(\mathbb{R}P^2, p)$

$1 = [c] = [f_2] = [f_1 * f_1] = [f_1]^2$

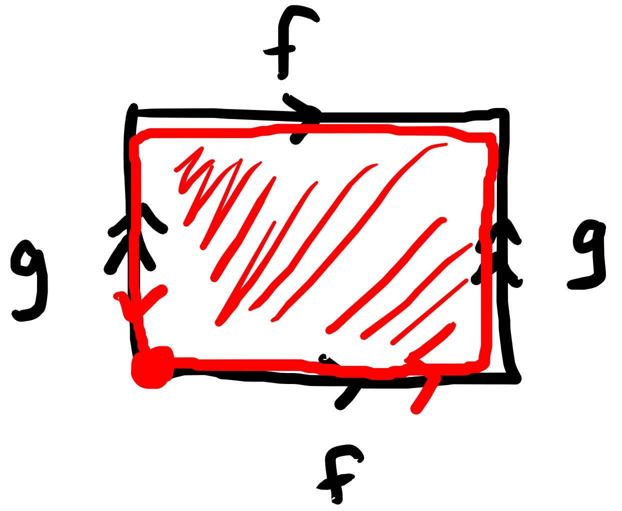
Example $T^2 = S^1 \times S^1$

$f, g : [0,1] \longrightarrow T^2$



$f(t) = (e^{2\pi i t}, 1) \qquad g(t) = (1, e^{2\pi i t})$

$f * g * \bar{f} * \bar{g} \sim c$



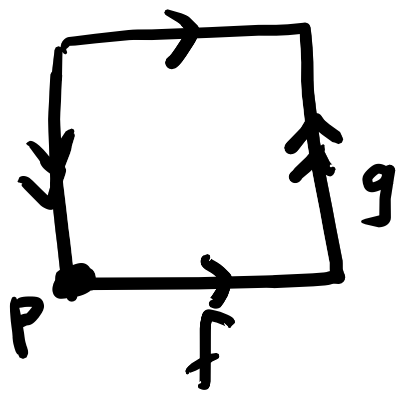
$\bar{f}(t) = f(1-t)$

Thus in $\pi_1(T, p)$

$[f]\,[g]\,[f]^{-1}\,[g]^{-1} = 1 \implies [f][g] = [g][f]$

$\implies [f]$ and $[g]$ commute.

OTOH for K = Klein bottle



it's not clear if $[f][g]=[g][f]$

since $f * g * \bar{f} * \bar{g}$ doesn't obviously extend to $\mathbb{D}^2$.

Instead, we see that $f * g * \bar{f} * g$ extends to $\mathbb{D}^2$

So in $\pi_1(K,p)$ have $[f][g][f]^{-1}[g]=1$.

$\Rightarrow [f][g]=[g]^{-1}[f]$.

<u>Rmk</u>. If $[f][g]=[g][f]$ then $[g][f]=[g]^{-1}[f] \Rightarrow [g]=[g]^{-1}$

$\Rightarrow [g]^2=1$.

later : show $[f], [g]$

have ∞ order.

Hence $[f], [g]$ don't commute.

$\Rightarrow \pi_1(K, p)$ is not abelian.