I. More quotient maps

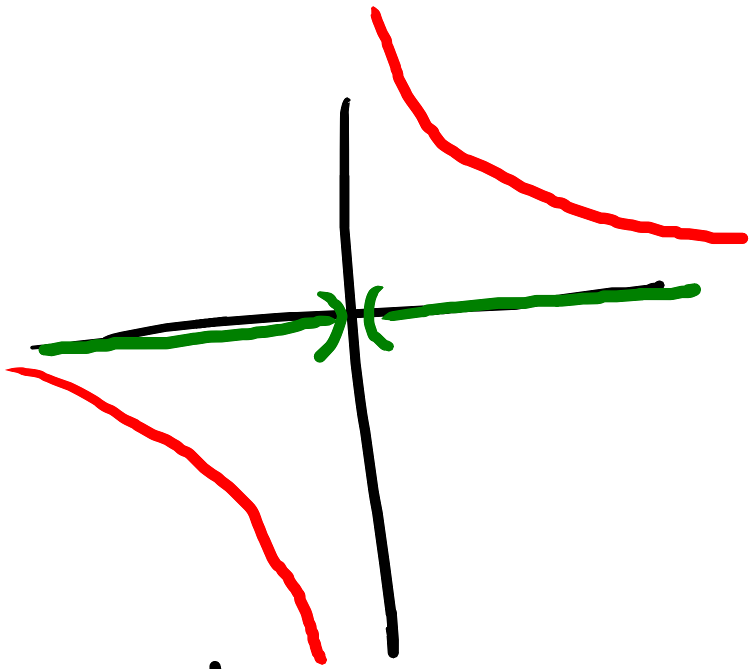
<u>Recall</u> A surjective map $q: X \to Y$
is a <u>quotient map</u> if $q^{-1}(U)$ open implies U open
for $U \subset Y$

<u>Q</u>: How to tell is a map is a quotient map

<u>Defn</u> Say $f: X \to Y$ is <u>open</u> if $U \subset X$ open $\Rightarrow f(U) \subset Y$ open
— " — <u>closed</u> if $A \subset X$ closed $\Rightarrow f(A) \subset Y$ closed

Ex $\quad p: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x$



- p is not closed

  $A = \{(x, \frac{1}{x}) : x \neq 0\} \subset \mathbb{R}^2$ closed

  $p(A) = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$ not closed

- p is open: suffices to check on basis of $\mathbb{R}^2$

  eg open rectangles $\quad p((a,b) \times (c,d)) = (a,b)$

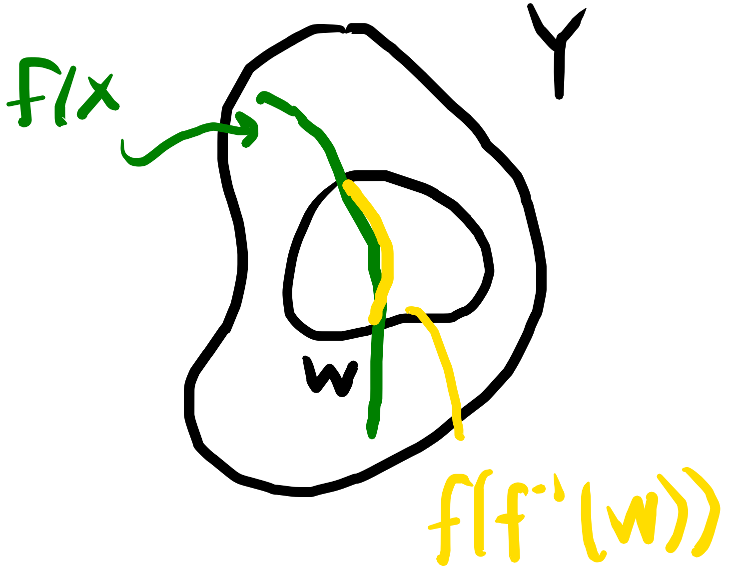
Prop Fix $f: X \to Y$ surj.

If f either open / closed then f quotient map.

Proof Assume f open

Fix $W \subset Y$ s.t. $f^{-1}(W)$ open (WTS W open)

f open $\Rightarrow$ $\underset{\overset{\shortparallel}{W}}{f(f^{-1}(W))}$ open (f surjective) $\square$



(case f closed $\to$ exercise)

Cor X compact, Y Hausdorff

$f: X \to Y$ surj $\Rightarrow$ f quotient

Proof By prop, suffices to show f closed. Fix $A \subset X$ closed

A closed $\Rightarrow$ A compact (X compact)

$\Rightarrow$ $f(A)$ compact

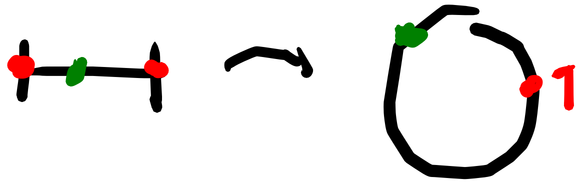
$\Rightarrow$ $f(A)$ closed (Y Hausdorff) $\square$

II. Identifying quotient spaces

X space, $\mathcal{P}$ partition, $\pi : X \to \mathcal{P}$

Quotient top on $\mathcal{P}$: $U \subset \mathcal{P}$ open if $\pi^{-1}(U) \subset X$ open.

<u>Thm</u> (last time) $q : X \to Y$ quotient map



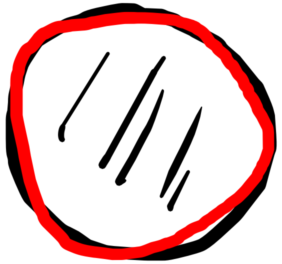
$$\mathcal{P} = \{ q^{-1}(y) : y \in Y \}. \text{ Then } \mathcal{P} \cong Y.$$

<u>Applications</u>

(1) $X = [0, 2\pi]$ $\quad \mathcal{P}$: $\{x\}$ $\;x \in (0, 2\pi)$; $\{0, 2\pi\}$ $\quad$ <u>Claim</u> $\mathcal{P} \cong S^1$

<u>Pf</u> consider $X \xrightarrow{q} S^1$ $\quad q(x) = e^{ix}$ quotient. associated Partition of q is $\mathcal{P}$ ✓

(2) $X = \mathbb{D}^2 = \overline{B_1(0)} \subset \mathbb{R}^2$



$\mathcal{P}$: $\{x\}$ for $x \in B_1(0)$

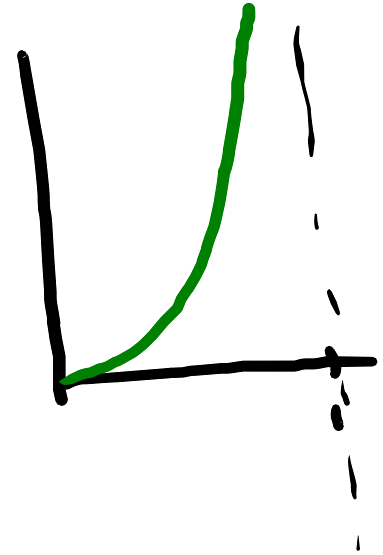
$X \setminus B_1(0) \cong S^1$

<u>Claim</u> $\mathcal{P} \cong S^2$

<u>Want</u>: quotient map $q: X \to S^2$ w/ associate partition $\mathcal{P}$.

<u>Step 1</u> $B_1(0) \cong \mathbb{R}^2$

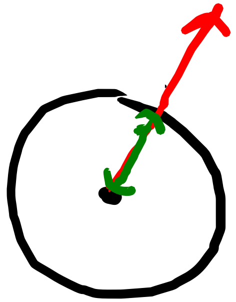
$(0,1) \cong (0,\infty)$



define $h: B_1(0) \longrightarrow \mathbb{R}^2 = \mathbb{C}$

$f(x) = \frac{x}{1-x}$

$h(re^{i\theta}) = f(r)e^{i\theta}$



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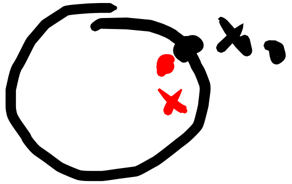


P : $\{x\}$ for $x \in B_1(0)$

$X \setminus B_1(0) \cong S^1$

Claim $P \cong S^2$

Step 1 $B_1(0) \cong \mathbb{R}^2$ $\quad h : B_1(0) \to \mathbb{R}^2$

Step 2 $\mathbb{R}^2 \cong S^2 \setminus \{\infty\}$ stereographic projection

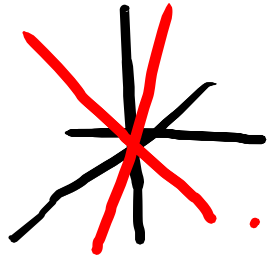
$g : \mathbb{R}^2 \longrightarrow S^2 \setminus \{\infty\}$.

Step 3 define $q : X \longrightarrow S^2$ $\quad x \longmapsto \begin{cases} g(h(x)) & x \in B_1(0) \\ \infty & \underline{x \in S^1} \end{cases}$

Ex q continuous. Partition of P: $q^{-1}(y)$ $\quad y \in S^2$ $\quad y = \infty$ $\quad q^{-1}(y) = S^1$ $\quad y \neq \infty$ $\quad q^{-1}(y) = pt$

II. Projective plane

$\mathbb{R}P^2 = \{\text{lines through } 0 \text{ in } \mathbb{R}^3\}$



To topologize, use quotient top.

$X = \mathbb{R}^3 \setminus \{0\}$ partition $P = \left\{ \{cv \mid c \in \mathbb{R}\setminus\{0\}\} : \begin{array}{l} v \in \mathbb{R}^3 \\ v \neq 0 \end{array} \right\}$

$\equiv \mathbb{R}P^2$

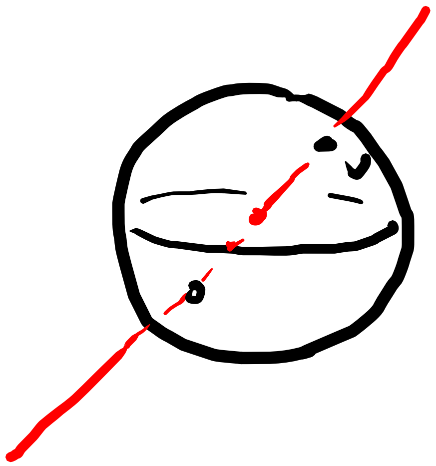
Notation: write $[x:y:z] \in \mathbb{R}P^2$ for line through (x,y,z)

$[ax:ay:az] = [x:y:z]$ for any $a \neq 0$.

Alternate definitions

(1) There's a surjection (quotient map)

$$S^2 \longrightarrow \mathbb{R}P^2$$
$$v \longmapsto \mathrm{span}(v)$$

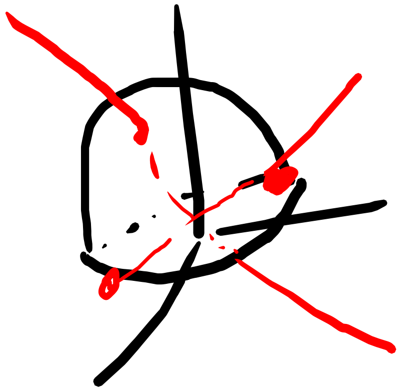


induced partition of S^2

$\mathcal{P}$: $\{v, -v\}$ $v \in S^2$

So $\mathbb{R}P^2$ also described as quotient of S^2 obtained by identifying antipodal points $v \longleftrightarrow -v$.

② $X = \{(x,y,z) \in S^2 : z \geq 0\} \cong \mathbb{D}^2$



There is a surj $X \xrightarrow{q} \mathbb{R}P^2$

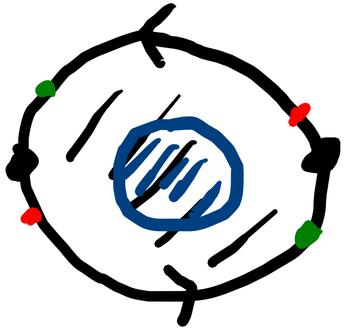
$v \longmapsto \mathrm{span}(v)$

The associated partition of $\mathbb{D}^2$

$\mathcal{P}$: $\{(x,y)\}$ $x^2+y^2<1$.

$\{(x,y),(-x,-y)\}$ $x^2+y^2=1$

$\mathbb{R}P^2$ is quotient of $\mathbb{D}^2$:



$= \mathbb{D}^2 \cup$ Möbius

$= N_1$ (HW3)

III Group actions

<u>Defn</u> G group, X space

A <u>group action</u> is a homomorphism $\phi : G \longrightarrow \mathrm{Top}(X)$

ie for each $g \in G$ have top. equiv $\phi(g) : X \longrightarrow X$

homomorphism $\phi(g \circ h) = \phi(g) \circ \phi(h)$, implies $\phi(e) = id_X$

Can view a group action as a multiplication of sorts

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x := \phi(g)(x) \end{aligned}$$

<u>Rmk</u> If G top. group then also want ϕ continuous

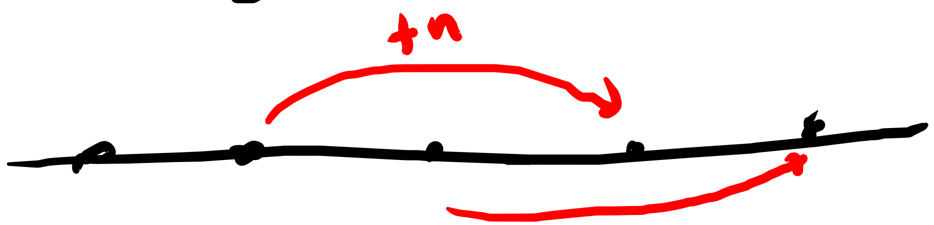
Examples

- $X = \{1, \dots, n\}$ w/ discrete top.

  $\mathrm{Top}(X) = S_n$ symmetric group (bijections $\{1,..,n\} \circlearrowleft$) aka permutation

- $G = \mathbb{Z}$ acts on $X = \mathbb{R}$ by translations

  $n \cdot x = x + n$



- $G = GL_n\mathbb{R}$ acts on $X = \mathbb{R}^n$ by linear maps

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

- $G = O(n)$ acts $X = S^{n-1} = \{v \in \mathbb{R}^n : |v| = 1\}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$