I. Quotient spaces "spaces obtained by gluing"

Defn A *partition* $\mathcal{P}$ on a set/space X is a collection of disjoint subsets of X whose union is X.

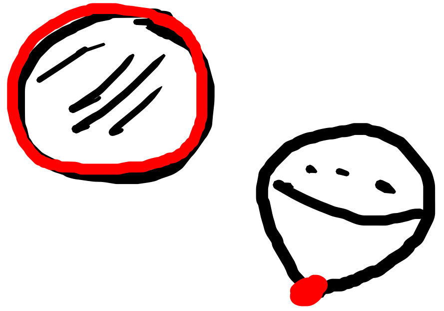
$(\mathcal{P} \subset P(X))$

Examples (1) "circle" $X = [0,1]$

$\mathcal{P}$ partition w/ elements $\{x\}$ $x \in (0,1)$

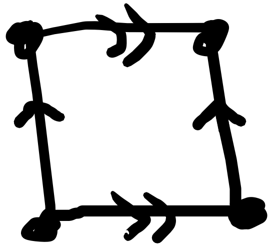
$\{0,1\}$

(2) "sphere" $X = D^2 = \overline{B_1(0)} \subset \mathbb{R}^2$



$\mathcal{P}$: $\{x\}$ for x w/ $|x|<1$ $\{x : |x| = 1\}$

(3) "torus"



$X = [0,1] \times [0,1]$

$\mathcal{P}$:

$\{(x,y)\}$ $\quad x, y \in (0,1)$

$\{(0,y), (1,y)\}$ $\quad y \in (0,1)$

$\{(x,0), (x,1)\}$ $\quad x \in (0,1)$

$\{(0,0), (1,0), (0,1), (1,1)\}$

(4) $f: X \longrightarrow Y$ surjective

$\mathcal{P} = \{ f^{-1}(y) : y \in Y \}$ is a partition

Given $X, \mathcal{P}$ there is a map $\pi : X \longrightarrow \mathcal{P}$

$\pi(x)$ = unique element of $\mathcal{P}$ containing x.

<u>Defn</u> The quotient topology on $\mathcal{P}$

$U \subset \mathcal{P}$ open if $\pi^{-1}(U) \subset X$ is open.

- check this is a topology
  - $\phi, \mathcal{P}$ open
  - unions $U_\alpha \subset \mathcal{P}$ open, $\pi^{-1}(\bigcup U_\alpha) = \bigcup \underbrace{\pi^{-1}(U_\alpha)}_{\text{open}} \subset X^{\text{open}}$

    $\Rightarrow \bigcup U_\alpha$ open.
  - intersections: similar

Rmk By construction, $\pi : X \to \mathcal{P}$ is continuous. Moreover the quotient topology on $\mathcal{P}$ is the largest top. w/ this property.

Ex $X = [0,1]$ $\quad \mathcal{P}$ as above

Claim $\mathcal{P} \cong S^1$.

Note there are maps

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\exp"] & \\ \mathcal{P} \arrow[rr, dashed, "h"'] & & S^1 \end{tikzcd}$$

$$\exp(t) = e^{2\pi i t} \in \mathbb{C}$$
$$= (\cos 2\pi t, \sin 2\pi t) \in \mathbb{R}^2$$

Define $h(\{t\}) = e^{2\pi i t}$

$h(\{0,1\}) = 1$

we'll show h is a top equiv.

II Quotient maps

Lemma (continuity for quotient spaces)

X space, P partition. $f : P \to Z$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ P \arrow[r, "f"'] & Z \end{tikzcd}$$

Then f continuous $\iff$ $f\circ\pi$ continuous

Proof $(\Rightarrow)$ composition of cts.

$(\Leftarrow)$ Suppose $f\circ\pi$ cts. Fix $U \subset Z$ open. (WTS $f^{-1}(U) \subset P$ open)

$f^{-1}(U)$ open $\iff$ $\underbrace{\pi^{-1}(f^{-1}(U))}_{= (f\circ\pi)^{-1}(U)} \subset X$ open   holds b/c $f\circ\pi$ cts.

$\square$

Defn X, Y spaces

$q: X \to Y$ is a quotient map if q is surjective

and $U \subset Y$ is open whenever $q^{-1}(U) \subset X$ open.

ie q sends open sets of form $q^{-1}(U)$ to open sets

$$q(q^{-1}(U)) = U \quad (\text{when } q \text{ surj})$$

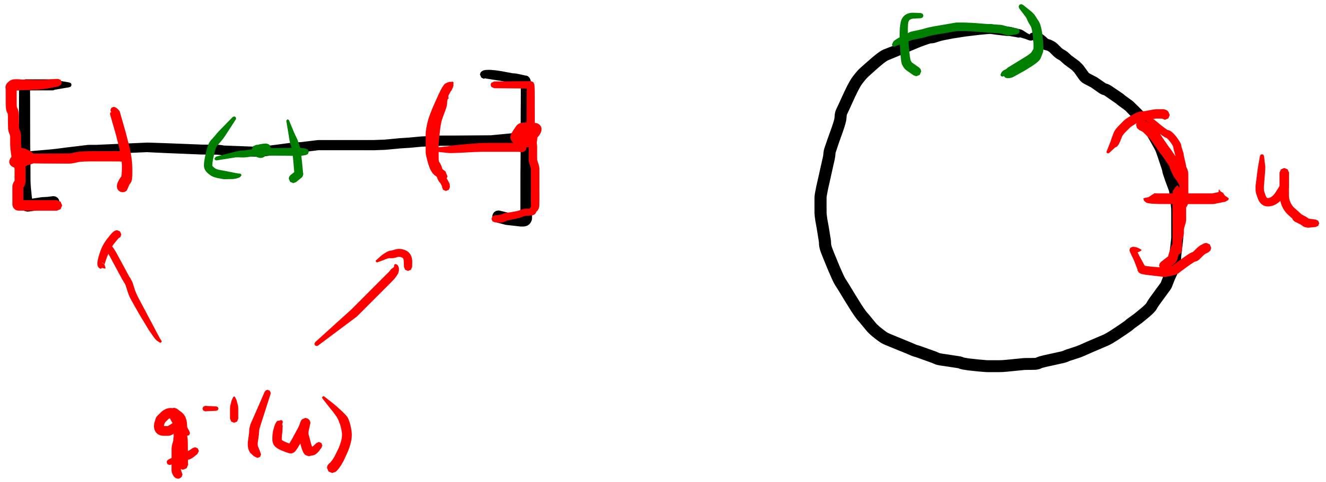
Rmk in general a cts map need not send open sets to open sets eg $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

$(-1,1)$ open $\quad f((-1,1)) = [0,1)$ not open

Ex $[0,1] \longrightarrow S^1$ is a quotient map

$t \longmapsto e^{2\pi i t}$

although it is not an open map



Lemma $X \xrightarrow{q} Y$ quotient map.

$f : Y \to Z$ cts $\iff$ $f \circ q$ cts

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f\circ q"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Pf Same as before. $\square$

<u>Thm</u> $X \xrightarrow{q} Y$ quotient map

$\mathcal{P} = \{q^{-1}(y) : y \in Y\}$ Then $\mathcal{P} \cong Y$

Corollary: given X w/ partition $\mathcal{P}$. To show $\mathcal{P} \cong Y$

suffices to find quotient map $X \xrightarrow{q} Y$ whose

associated partition is $\mathcal{P}$.

Thm $X \xrightarrow{q} Y$ quotient map

$\mathcal{P} = \{ q^{-1}(y) : y \in Y \}$ Then $\mathcal{P} \cong Y$

Proof Define

$$\begin{tikzcd} \mathcal{P} \arrow[r, bend left, "h"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$g(y) = q^{-1}(y) \in \mathcal{P}$

$h(q^{-1}(y)) = y$

inverses

WTS $\underline{h}, g$ continuous

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "h \circ \pi = q"] & \\ \mathcal{P} \arrow[r, "h"'] & Y \end{tikzcd}$$

by lemma h cts $\Leftrightarrow$ $h \circ \pi$ cts. $\Leftrightarrow$ q cts ✓

$h(\pi(x)) = h\left(q^{-1}(q(x))\right) = q(x)$ by defn

$\square$

III Topological groups

Defn A group is a set w/ a multiplication

$G \times G \longrightarrow G$ that is • associative $(ab)c = a(bc)$

$(a,b) \longmapsto a \cdot b$ • have identity e, $ea = ae = a$

• inverses, $\forall a \; \exists b$, $ab = ba = e$.

eg $(\mathbb{Z}, +)$ $(\mathbb{R}, +)$ $(\mathbb{R} \setminus \{0\}, \cdot)$

Defn A group G w/ a topology is a topological group

if mult $G \times G \longrightarrow G$ and inversion $G \to G$ $(a \mapsto a^{-1})$

are continuous.

Examples

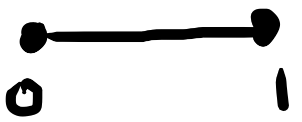
- $(\mathbb{R}, +)$ $\quad$ $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $(a,b) \longmapsto a+b$ $\quad$ $\mathbb{R} \longrightarrow \mathbb{R}$, $a \longmapsto -a$

- $(\mathbb{Z}, +)$

- quotient group $\mathbb{R}/\mathbb{Z}$ is a topological group w/ quotient top.

$$\mathbb{R}/\mathbb{Z} = \{ \mathbb{Z} + a \mid a \in \mathbb{R} \} \longleftarrow \text{partition of } \mathbb{R}$$

$$= \{ \mathbb{Z} + a \mid a \in [0,1] \} \qquad \mathbb{Z} + 0 = \mathbb{Z} + 1$$



$$\mathbb{R}/\mathbb{Z} \cong S^1$$

- $GL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) \neq 0 \} \subset M_n\mathbb{R} \cong \mathbb{R}^{n^2}$

$\cup$

$SL_n(\mathbb{R}) = \{ \det(A) = 1 \}$

$\cup$

$SO(n) = \left\{ \begin{array}{l} \det(A) = 1 \\ A^t A = I \end{array} \right\}$

- $Top(X) = \{ f : X \to X \text{ top. equivalence} \}$

mult $\leftrightarrow$ composition

topological group w/ "compact-open topology" ; $K \subset X$ cpt, $U \subset X$ open

$N(K, U) = \{ f(K) \subset U \}$