I. Path connectedness

Last time: X disconnected if

$X = U \cup V$ $\quad U, V$ nonempty, disjoint, open

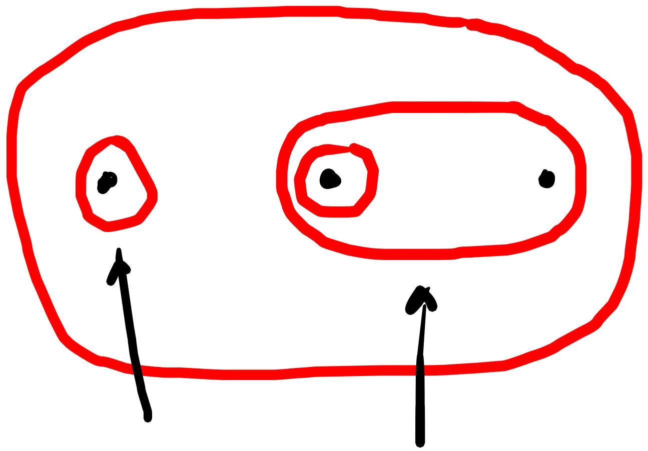
Warmup: X connected $\Longleftrightarrow$ the only clopen subsets of X are ϕ, X

equivalently X disconnected $\Longleftrightarrow$ $\exists$ $U \subset X$ clopen
$U \neq \phi, X$

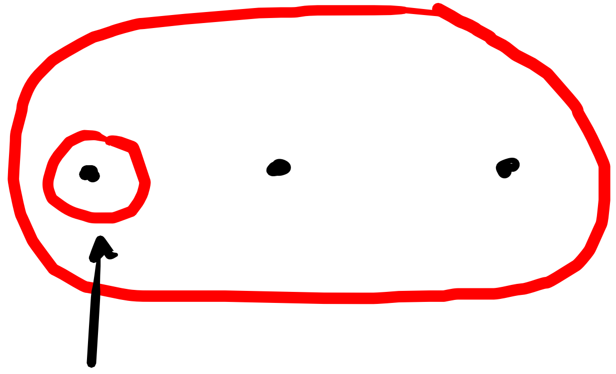
$X = U \cup V$

- we proved $\mathbb{R}$ connected $\Rightarrow$ only clopens in $\mathbb{R}$ are $\phi, \mathbb{R}$.

Ex.



disconnected

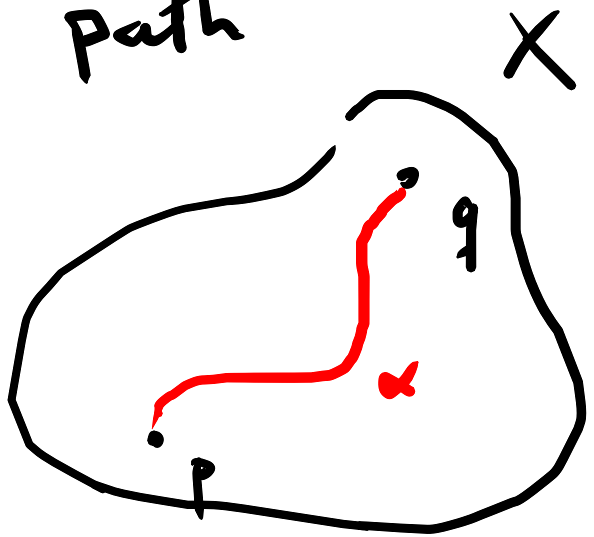


connected.

Defn A <u>path</u> is a continuous map $\alpha : [0,1] \to X$

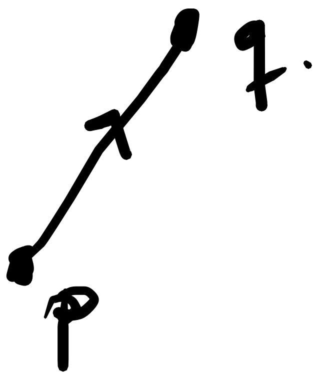
Say X is path connected if $\forall\, p, q \in X$ $\exists$ path

$\alpha : [0,1] \to X$ s.t. $\alpha(0) = p$ $\alpha(1) = q$



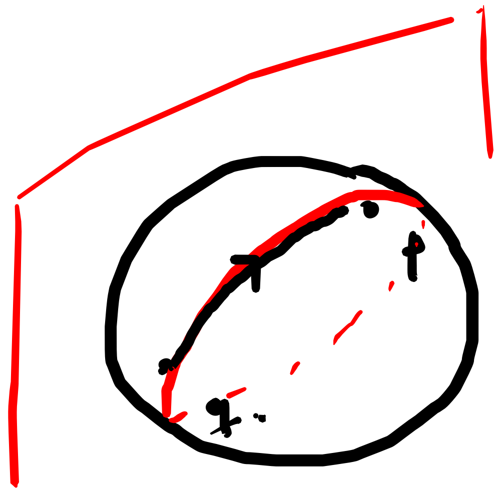
Examples

- $\mathbb{R}^n$ is path connected.



$$\alpha(t) = p + t(q-p)$$

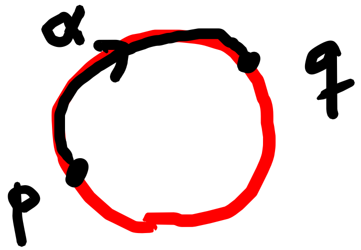
- $S^n = \{ z \in \mathbb{R}^{n+1} : |z| = 1 \}$



path connected. Fix $p, q \in S^n$

Consider plane $V = \text{span}\{p, q\} \cong \mathbb{R}^2$

$V \cap S^n \cong S^1$



Choose arc on $V \cap S^n \cong S^1$.

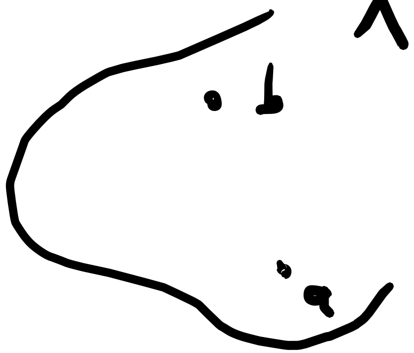
Prop X path connected $\Rightarrow$ X connected

Proof By contradiction, suppose $X = U \cup V$.
Take $u \in U$, $v \in V$, and $\alpha : [0,1] \longrightarrow X$ path from u to v. $[0,1] = \underset{\ni\, 0}{\alpha^{-1}(U)} \cup \underset{\ni\, 1}{\alpha^{-1}(V)}$

$\Rightarrow$ $[0,1]$ disconnected $\ast$. $\square$

Rmk Path connected is a topological property:

X



$X \xrightarrow{f} Y$ Y path conn $\rightarrow$ X path conn.

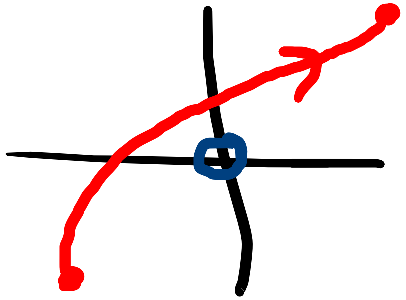
Application $\mathbb{R} \not\cong \mathbb{R}^2$.

Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ top. equiv.

Set $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ $\quad g(x) = f(x) - f(0) \quad g(0) = 0$.

Then g restricts to $g| : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$ top. equiv

- OTOH $\mathbb{R}\setminus\{0\}$ not path connected (not connected)
- OTOH $\mathbb{R}^2\setminus\{0\}$ is path connected

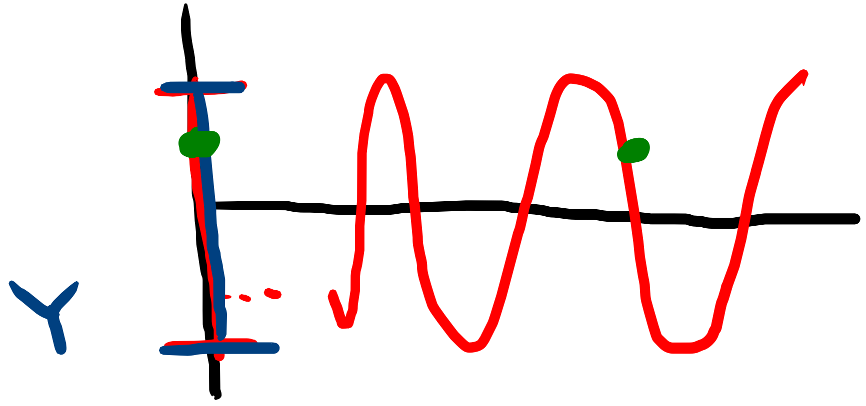


$\times$.

Therefore $\mathbb{R} \not\cong \mathbb{R}^2$.

Ex Topologist sine curve X

Prop X is not path connected.



Last time showed X connected

path conn $\Longrightarrow$ conn

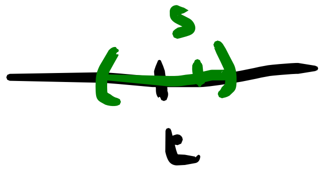
$\cancel{\Longleftarrow}$

Sketch of Prop $Y := \{0\} \times [-1,1]$. Let $\alpha : [0,1] \to X$ with $\alpha(0) \in Y$.

Claim. $\alpha([0,1]) \subset Y$.

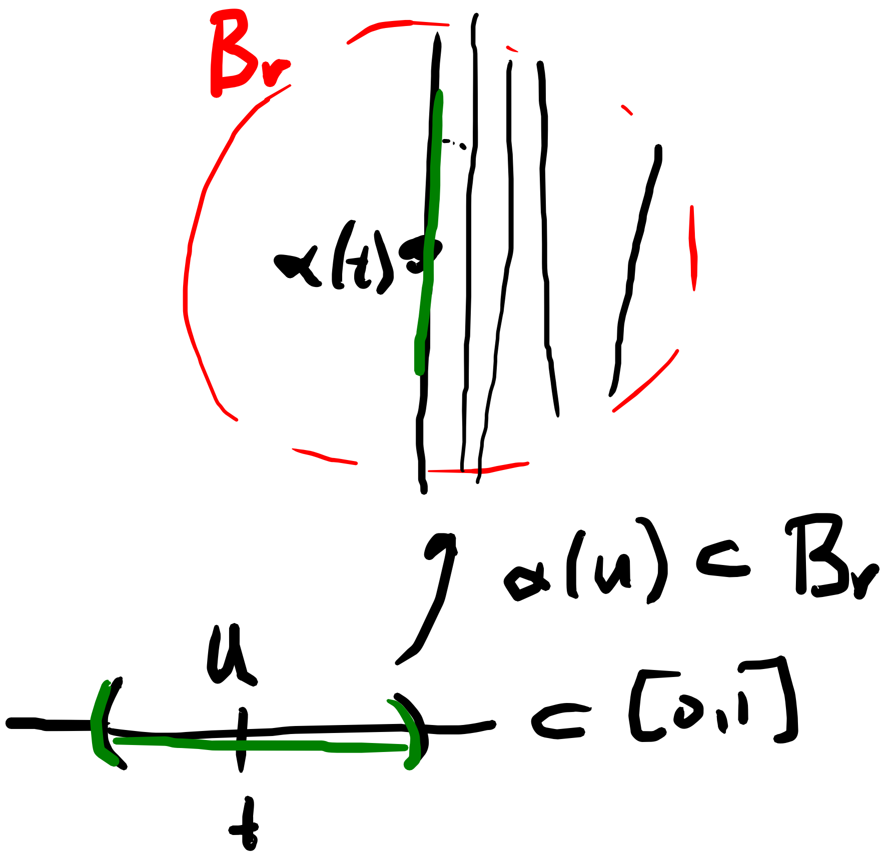
To prove: show $0 \in \alpha^{-1}(Y) \subset [0,1]$ clopen.

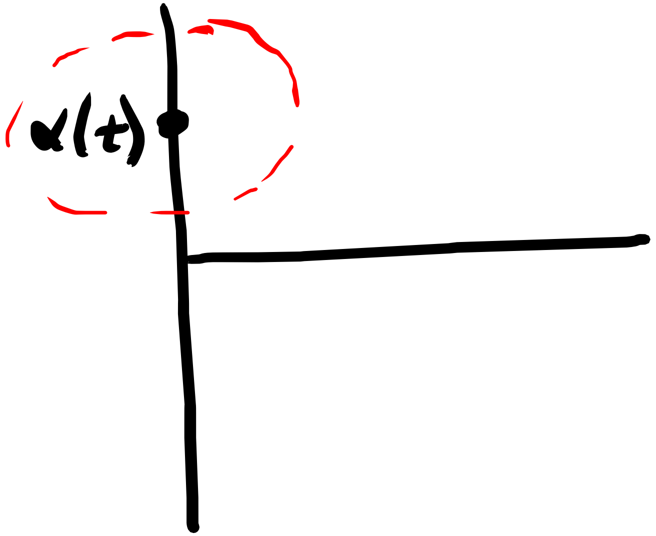
- $\alpha^{-1}(Y)$ closed b/c Y closed
- $\alpha^{-1}(Y)$ open: fix $t \in [0,1]$ $\alpha(t) \in Y$.



$\exists$ ball B_r around $\alpha(t)$ s.t. $B_r \cap X$

is a disjoint union of arcs



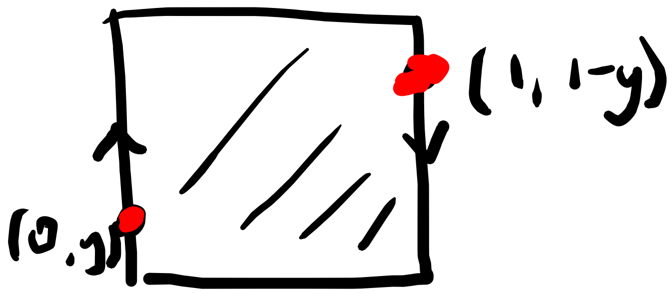


II. Quotient spaces

Rigorous definition of Möbius band (as a topological space)

$X = [0,1] \times [0,1]$

define equivalence relation $(0,y) \sim (1, 1-y)$. All other points are in own equiv. class.



$M = X/\sim$ set of equivalence class.

want topology on M.

so that the natural map $X \xrightarrow{P} X/\sim = M$ is continuous.

Force this: by declaring

$V \subset M$ open if $p^{-1}(V) \subset X$ open.

Need to check that this defines a topology ...