Homework 8

Math 25b

Due April 19, 2018

Topics covered: manifolds, Lagrange multipliers, differential forms

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Davis L.

Problem 1. *Show that the set $X \subset \mathbb{R}^3$ given by $x^3 + xy^2 + yz^2 + z^3 = 4$ is a 2-dimensional manifold.*[1]

Solution. □

Problem 2. *What is the volume of the largest rectangular parallelepiped contained in the ellipsoid $x^2 + 4y^2 + 9z^2 \leq 9$?*

Solution. □

Problem 3.

(a) *Recall/consider $SL_2(\mathbb{R}) \subset M_2(\mathbb{R})$, the subset of matrices with determinant 1. Show that $SL_2(\mathbb{R})$ is a 3-manifold.* [2]

(b) *Find a matrix in $SL_2(\mathbb{R})$ that is closest to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hint: avoid minimizing a function with square roots. There is more than one matrix that works; don't try to find all of them.*

Solution. □

[1] Sometimes we shorten "2-dimensional manifold" to "2-manifold". This convention appears in later problems.

[2] Matrix multiplication makes $\mathrm{SL}_2(\mathbb{R})$ into a group(!). A group that is also a manifold is called a *Lie group*, named after Sophus Lie.

2 For Joey F.

Problem 4.

(a) For what values of a and b are the sets $X_a = \{(x, y, z) : x - y^2 = a\}$ and $Y_b = \{(x, y, z) : x^2 + y^2 + z^2 = b\}$ 2-manifolds in $\mathbb{R}^3$?

(b) For what values of a and b is the intersection $X_a \cap Y_b$ a 1-manifold? What geometric relation is there between X_a and Y_b for other values of a and b?

Solution. □

Problem 5.

(a) Fix a vector $w \in \mathbb{R}^3$, and define a 2-form $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi(u, v) = \det(u, v, w).$$

Express ϕ as a linear combination of elementary forms.

(b) Fix vectors $u, v \in \mathbb{R}^3$ and define a 1-form $\psi : \mathbb{R}^3 \to \mathbb{R}$ by

$$\psi(w) = \det(u, v, w).$$

Express ψ as a linear combination of elementary forms. What is the connection with the cross product $u \times v \in \mathbb{R}^3$?

Solution. □

Problem 6. *Answer each statement true or false, and give an explanation.*

(a) If $\omega_1, \omega_2 \in \Lambda^1(\mathbb{R}^3)$ and $\eta \in \Lambda^2(\mathbb{R}^3)$ are nonzero and $\omega_1 \wedge \eta = \omega_2 \wedge \eta$, then $\omega_1 = \omega_2$.

(b) If $\omega_1, \omega_2 \in \Lambda^1(\mathbb{R}^3)$ and $\omega_1 \wedge \eta = \omega_2 \wedge \eta$ for all $\eta \in \Lambda^2(\mathbb{R}^3)$, then $\omega_1 = \omega_2$.

Solution. □

3 For Laura Z.

Problem 7. *Write each of the following 9-forms on $\mathbb{R}^{13}$ as an elementary form*

(a) $dx_7 \wedge dx_{10} \wedge dx_6 \wedge dx_9 \wedge dx_3 \wedge dx_{13} \wedge dx_4 \wedge dx_5 \wedge dx_2$

(b) $dx_6 \wedge dx_{11} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_8 \wedge dx_{10} \wedge dx_5 \wedge dx_9$

(c) $dx_{12} \wedge dx_2 \wedge dx_9 \wedge dx_8 \wedge dx_3 \wedge dx_2 \wedge dx_5 \wedge dx_1 \wedge dx_7$

Solution. □

Problem 8. *Let $f \in \Lambda^k(\mathbb{R}^n)$ and let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Define $T^*f : V \to \mathbb{R}$ by $T^*f(v_1, \ldots, v_k) = f(Tv_1, \ldots, Tv_k)$.*

(a) *Show that T^* defines a linear map $\Lambda^k(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^m)$.* [3]

(b) *Assume $m = n$. Show that $T^*(dx_1 \wedge \cdots \wedge dx_n)$ is $\det T \cdot dx_1 \wedge \cdots \wedge dx_n$.*

Solution. □

Problem 9. *Consider the differential form $\omega = x\,dx + y\,dy \in \Omega^1(\mathbb{R}^2)$.*

(a) *Compute $\omega(p)(v)$ for $p = (-1, 2)$ and $v = (4, 3)$.*

(b) *Find $p \neq q$ so that $\omega(p)$ and $\omega(q)$ are linearly dependent in $\Lambda^1(\mathbb{R}^2)$.*

(c) *Let F be the vector field associated to ω (as discussed in class). Compute the line integrals $\int_0^1 F(c(t)) \cdot c'(t)\,dt$ for $c_1(t) = (\cos(2\pi t), \sin(2\pi t))$ and $c_2(t) = (1 + t, 0)$.*

Solution. □

[3] For $f \in \Lambda^k(\mathbb{R}^n)$, the form T^*f is called the *pullback* of f along T.

4 For Beckham M.

Problem 10. *Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. In this problem you prove A has an eigenvector.* [4]

(a) *Define $\phi : \mathbb{R}^n \to \mathbb{R}$ by $\phi(x) = \langle x, Ax \rangle = x^t Ax$ (the angle brackets denote the standard inner product). Show ϕ is C^1 and prove the formula*

$$D\phi(x)(u) = 2x^t Au.$$

Hint: you'll need to use that A is symmetric, potentially more than once.

(b) *Prove that there exists a maximum of ϕ restricted to the unit ball $X = \{x \in \mathbb{R}^n : |x|^1 = 1\}$. Hint: you will need to use a theorem from way back at the beginning of the course.*

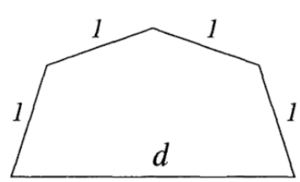
(c) *Use Lagrange's theorem to conclude that A has an eigenvector.*

Solution. □

Problem 11. *Prove the configuration of two linked rods with one endpoint fixed is a manifold using the manifold recognition theorem.* [5]

Solution. □

Problem 12. *Consider linked rods made of four unit segments connecting fixed points located at distance $d \leq 4$. See the figure below.*



(a) *Fix $d = 3.9$, and let M be the configuration space. Use manifold recognition to prove that M is a manifold. What is its dimension?*

(b) *Show M is topologically equivalent to the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Hint: you can do this without any algebra. It may help to first understand the case of 3 linked rods.*

Solution. □

[4] This is the hard part of proving the spectral theorem. Once one finds a single eigenvector, an inductive argument shows that there is an orthonormal basis of eigenvectors.

[5] This is the example we discussed in class.

5 Extra credit

Problem 13. *This problem is extra credit, worth* 12 *points* [6].

(a) *Build a working model of the linked rods in Problem* 12*. Use whatever materials you like. You may collaborate on this part if you want.*

(b) *Determine the topology of the configuration space when* $d = 1$*. (It's a sphere with some number of handles – how many?) Hint: you can do this without any algebra.*

(c) *For which values of* $d \in (0, 4)$ *is* M *not a manifold?*

Solution. □

[6] + my admiration