Homework 7

Math 25b

Due April 12, 2018

Topics covered: multivariable derivative, differentiability, partial derivatives, chain rule, continuous partials theorem, examples related to the inverse function theorem

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Joey F.

Problem 1. *Find $Df(x, y)$ for the following functions.*

(a) $f(x, y) = \sin(x \sin y)$

(b) $f(x, y) = \sin(xy)$

(c) $f(x, y) = (\sin(xy), \sin(x \sin y))$

Solution. □

Problem 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(0) = 0$ and*

$$f(x, y) = xy/(x^2 + y^2) \quad \text{if } (x, y) \neq 0.$$

(a) *For which $u \neq 0$ does the directional derivative $D_u f(0)$ exist? Compute it when it exists.*

(b) *Is f differentiable at 0?*

(c) *Is f continuous at 0? Hint: it can be helpful to use polar coordinates.*

Solution. □

Problem 3. *Repeat the previous problem for the function defined by $f(0) = 0$ and*

$$f(x, y) = x^3y^2/(x^2y^2 + (y + x)^2) \quad \text{if } (x, y) \neq 0.$$

Solution. □

2 For Laura Z.

Problem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function with coordinate functions $f_1, \dots, f_m$. Show that f is differentiable if and only if each f_i is, and in that case $Df(a)$ is the matrix whose i-th row is $Df_i(a)$. Hint: you need to use the definition of the derivative directly for this problem.* [1]

Solution. □

Problem 5. *Define $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $J(x, y) = \langle x, y \rangle$ (the standard inner product).*

(a) *Find $DJ(a, b)$.*

(b) *If $f, g : \mathbb{R} \to \mathbb{R}^n$ are differentiable and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t) \rangle$, show that*

$$h'(a) = \langle g(a), f'(a) \rangle + \langle f(a), g'(a) \rangle.$$

(c) *If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $|f(t)| = 1$ for all t, show that $\langle f(t), f'(t) \rangle = 0$.* [2]

Solution. □

Problem 6. *Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$.*

(a) *Show that $Df(x, y)$ is invertible for every (x, y).*

(b) *Show that f is not injective.* [3]

Solution. □

[1] By this problem, to determine if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, it suffices to ask the same question for the coordinate functions $f_i : \mathbb{R}^n \to \mathbb{R}$. If $n = 1$, this means studying functions of one variable. Hence all of the additional difficulty/subtlety with wording with derivatives in higher dimensions occurs when the *domain* has dimension ≥ 2.

[2] This exercise gives information about the *tangent space* of the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. (This will be discussed later.)

[3] This problem is related to a theorem we'll prove next week: The inverse function theorem says that a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ with invertible derivative is locally injective and f^{-1} is differentiable. The example of this problem shows that this statement is false if you remove the word "locally".

3 For Beckham M.

Problem 7. *Let $s : \mathbb{R}^2 \to \mathbb{R}$ be the "sum" function $s(x, y) = x + y$, and let $p : \mathbb{R}^2 \to \mathbb{R}$ be the "product" function $p(x, y) = xy$.*

(a) Show that s and p are differentiable and compute their derivatives.

(b) Use (a) together with the chain rule to prove the sum and product formulas for derivatives of functions $f, g : \mathbb{R}^n \to \mathbb{R}$.

Solution. □

Problem 8. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by the equation $f(x) = |x|^2 \cdot x$. Show that f has derivative of all orders and that f carries the unit ball $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ to itself in an injective fashion. Show however that the inverse function is not differentiable at 0.*

Solution. □

Problem 9. *Let $f : (a, b) \to \mathbb{R}$ be C^1 and $f'(x) \neq 0$ for all $x \in (a, b)$. In this problem you prove the 1-dimensional case of the inverse function theorem.*

(a) Prove that f is either increasing or decreasing, and conclude that f is injective.

(b) Prove directly[4] that the inverse function $g : f((a, b)) \to (a, b)$ is continuous.

(c) Show that g is differentiable. Hint 1: you can't use the chain rule (why?). Hint 2: Start with the expression $\frac{g(y+k)-g(y)}{k}$, define $h(k) = g(y + k) - g(y)$, and re-write $k = (y + k) - y = f(g(y + k)) - f(g(y)) = f(g(y) + h(k)) - f(g(y))$.

Solution. □

[4] You could use (c) to conclude that g is continuous, but don't... In fact you will need (b) to prove (c).

4 For Davis L.

Problem 10. *In class we proved the continuous partials theorem: if $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that the partial derivatives $D_i f$ exist and are continuous on a neighborhood of $a \in \mathbb{R}^n$, then f is differentiable at a. Show that the proof goes through if we assume only that each $D_i f$ exists in a neighborhood of a and is continuous at a.*

Solution. □

Problem 11. *Regard an $n \times n$ matrix as an element of the n-fold product $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ by considering each column as a vector in $\mathbb{R}^n$.*

(a) Prove that $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ *is differentiable and*

$$D(\det)(a_1, \ldots, a_n)(x_1, \ldots, x_n) = \sum_{i=1}^{n} \det\big(a_1|\cdots|x_i|\cdots|a_n\big).$$

(b) If $a_{ij} : \mathbb{R} \to \mathbb{R}$ are differentiable functions for $1 \le i, j \le n$ and $f(t) = \det(a_{ij}(t))$, show that

$$f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t) & \cdots & a'_{1j}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n1}(t) & \cdots & a'_{nj}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

Solution. □

Problem 12. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Show that f is differentiable but not C^1.

(b) Show that f is not injective in any neighborhood of 0. Hint: this is the tricky part. [5]

Solution. □

[5] This problem is related to the inverse function theorem (see previous footnote). It shows that the C^1 hypothesis cannot be dropped.