Homework 4

Math 25b

Due March 8, 2018

Topics covered: integrability, fundamental theorem of calculus, measure

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one document.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Davis L.

Problem 1. *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Show that f *is integrable and* $\int_{I^2} f = \frac{1}{2}$.

Solution. □

Problem 2. *Look up L'Hopital's rule (Pugh Ch. 3, Thm. 7). Use it to give an alternate proof of the first part of the approximation theorem: if* P *is the* k*-th order Taylor polynomial of* f *at* a*, then* $\lim_{h\to 0} \frac{f(a+h)-P(a+h)}{h^k} = 0$.

Solution. □

Problem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous. Show that the graph* $G_f = \{(x, f(x)) : x \in [a,b]\}$ *has measure 0 in* $\mathbb{R}^2$. *Hint: use uniform continuity.*

Solution. □

2 For Joey F.

Problem 4 (Pugh 3.29)**.** *Prove that the interval $[a, b]$ does not have measure zero (or in Pugh's terminology, is not a zero set).*

(a) *Explain why the following observation is not a solution to the problem: "Every open interval that contains $[a, b]$ has length $> b - a$."*

(b) *Instead prove by $[a, b]$ is a not a zero set by contradiction. Hint: suppose there is a "bad" covering of $[a, b]$ by open intervals whose total length is $< b - a$. Without loss of generality the covering is finite (why?). Take a minimal bad cover $\mathcal{B} = \{I_1, \ldots, I_n\}$. To reach a contradiction, show there exists a bad covering with $n - 1$ elements.*

Solution. □

Problem 5 (Pugh 3.30)**.** *The middle-quarters Cantor set Q is formed by removing the middle quarter from $[0, 1]$, then removing the middle quarter from each of the remaining two intervals, then removing the middle quarter of the remaining four intervals, and so on.*

(a) *Prove that Q is a zero set.*

(b) *Formulate the natural definition of the middle β-Cantor set Q_β. Is Q_β always a zero set? Prove or disprove.*

Solution. □

Problem 6. *Let $C \subset \mathbb{R}^n$ be a bounded set contained in a closed rectangle Q. We say C is rectifiable if $\int_Q \chi_C$ exists. Prove that the unit ball $B_1(0) \subset \mathbb{R}^n$ is rectifiable.*

3 For Laura Z.

Problem 7. *Let $Q \subset \mathbb{R}^n$ be a closed rectangle, and assume $f : Q \to \mathbb{R}$ is integrable.*

(a) Show that if $m \leq f(x) \leq M$ for all $x \in Q$, then $m \cdot vol(Q) \leq \int_Q f \leq M \cdot vol(Q)$.

(b) Show that if $Q = [a, b]$, then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_{[a,x]} f$$

is continuous (even if f is not continuous!).

Solution. □

Problem 8 (Pugh 3.40)**.** *Set*

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{\pi}{x} & x > 0 \end{cases} \quad \textit{and} \quad g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

One of these functions has an antiderivative, and the other does not. Figure out which is which, and prove it.

Solution. □

Problem 9 (Pugh 3.27)**.** *In many calculus books, the definition of the integral is given as*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \frac{b-a}{n}$$

where x_k^* is the midpoint of the k-th interval of $[a, b]$ having length $(b - a)/n$, namely

$$[a + (k-1)(b-a)/n, a + k(b-a)/n].$$

See Stewart's Calculus*, for example.*

(a) If f is continuous, show that the calculus-style limit exists and equals the Riemann integral of f. Hint: this is a one-liner.

(b) Show by example that the calculus-style limit can exist for functions that are not Riemann integrable. For this reason, the calculus-style definition of the integral is inadequate for real analysis.

Solution. □

4 For Beckham M.

Problem 10. *Fix a closed rectangle $A \subset \mathbb{R}^n$. In this exercise, you show that the set of integrable functions $f : A \to \mathbb{R}$ has the structure of a vector space. Let $f, g : A \to \mathbb{R}$ be integrable.*

(a) *For any partition P of A and subrectangle S, show that*

$$m_S(f) + m_S(g) \leq m_S(f+g) \text{ and } M_S(f+g) \leq M_S(f) + M_S(g)$$

and therefore

$$L(f,P) + L(g,P) \leq L(f+g,P) \text{ and } U(f+g,P) \leq U(f,P) + U(g,P).$$

Hint: To show $m_S(f)+m_S(g) \leq M_S(f+g)$ is suffices to show $m_S(f)+m_S(g) \leq M_S(f+g)+\epsilon$ for every $\epsilon > 0$.

(b) *Show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.*

(c) *For any constant c, show that $\int_A cf = c\int_A f$.*

Solution. □

Problem 11. *Fix a closed rectangle $A \subset \mathbb{R}^n$. Show that if $f, g : A \to \mathbb{R}$ are integrable, so is $f \cdot g$.*

Solution. □

Problem 12 (Pugh 3.34)**.** *Assume $f : [a,b] \to \mathbb{R}$ is C^1. A critical point of f is an x such that $f'(x) = 0$. A critical value is any number y such that there exists a critical point x so that $y = f(x)$. Prove that the set of critical values is a zero set.*[1] *Equivalently, let $C = \{x \in [a,b] : f'(x) = 0\}$ be the set of critical points, and show that $f(C)$ has measure 0. Hint: You will need the full strength of f being C^1 rather than just differentiable. If f' is small on $(a-\delta, a+\delta)$, what does that say about the length of $f((a-\delta, a+\delta))$?*

Solution. □

[1] This is the "Morse–Sard Theorem" in dimension one.