Homework 3

Math 25b

Due February 22, 2018

Topics covered: compactness, continuity theorems, Heine–Borel, derivatives

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one document.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Joey F.

Problem 1. *Show that if $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are sequentially compact, then $X \times Y \subset \mathbb{R}^{n+m}$ is sequentially compact.*

Solution. □

Problem 2. *Assume that $X \subset \mathbb{R}^n$ is covering compact.*

(a) Show that X bounded. Hint: use compactness to cover X by finitely many balls of radius 1. Why does this imply that X is bounded?

(b) Show that X is closed. Hint: fix $y \in X^c$ and for each $x \in X$ choose $r > 0$ so that $y \notin B_r(x)$. Observe that this gives a cover of X, use compactness to obtain a finite subcover, and conclude.

Solution. □

Problem 3 (Pugh 2.88)**.**

(a) Prove that a closed subset A of a covering compact set $K \subset \mathbb{R}^n$ is covering compact. Hint: start with an open covering of A and add one open set to get a covering of K.

(b) Conclude that if $A \subset \mathbb{R}^n$ is closed and bounded, then A is covering compact. Hint: use that closed rectangles are compact together with (a). [1]

Solution. □

[1] This completes the proof that $X \subset \mathbb{R}^n$ is covering compact if and only if it is closed and bounded. Combined with class, it shows that covering and sequential compactness are equivalent notions.

2 For Laura Z.

Problem 4. *If U is open and $C \subset U$ is covering compact, show that there is a covering compact set D such that $C \subset \operatorname{int}(D)$ and $D \subset U$.*

Solution. $\square$

Problem 5. *In this problem you prove the onion ring theorem: Fix $n \geq 1$ and let*

$$Q_1 \supset Q_2 \supset \cdots$$

be a sequence of closed rectangles in $\mathbb{R}^n$. Then $\bigcap Q_k \neq \emptyset$.

(a) First prove this for $n = 1$. Hint: Write $Q_k = [a_k, b_k]$ and consider $z = \sup\{a_k\}$ (why does this least upper bound exist?).

(b) Observe that the general case follows from the case $n = 1$ (project to the coordinate axes).

Solution. $\square$

Problem 6. *Use the Onion Ring Theorem to give an alternate proof of the following statement of the Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ with $f(c) = 0$. Hint: Do this using the "method of bisection" similar to our proof that closed rectangles are covering compact.*

Solution. $\square$

3 For Beckham M.

Problem 7.

(a) *Explain why derivatives are a "local property": if $f(x) = g(x)$ for all x in some open interval containing a, then $f'(a) = g'(a)$.*

(b) *Find $f'(x)$ and $f''(x)$ for $f(x) = |x|^3$. Does $f'''(x)$ exist for all x?*

Solution. □

Problem 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Compute f' where it's defined. Show that f' is not continuous at 0. [2]

Solution. □

Problem 9. *Suppose that f is differentiable at 0 and $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g that is continuous at 0.*

Solution. □

[2] If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous, we call f continuously differentiable. In this problem you show that not all differentiable functions are continuously differentiable.

4 For Davis L.

Problem 10. *A sequence (x_n) in $\mathbb{R}^d$ is called Cauchy if for every $\epsilon > 0$ there exists $N > 0$ so that $n, m > N$ implies $|x_n - x_m| < \epsilon$. It's easy to check that a convergent sequence is Cauchy.*[3] *In this problem you prove the converse.*[4] *Fix a Cauchy sequence (x_n).*

(a) Show that every subsequence of (x_n) has a convergent subsequence. Hint: Bolzano–Weierstrass.

(b) Show that if (a_k) and (b_ℓ) are subsequences of (x_n) and $a_k \to a$ and $b_\ell \to b$, then $a = b$.

(c) Conclude from x_n converges. Hint: use a problem from HW2.

Solution. □

Problem 11. *Suppose that (a_n) is a Cauchy sequence and (a_{n_k}) is a subsequence that converges to b. Prove that (a_n) converges to b.*

Solution. □

Problem 12. *Explain why the following sets are compact.*

(a) The Sierpinski triangle $\mathcal{S} \subset \mathbb{R}^2$. (You may want to read more about the construction of the Sierpinski triangle on Wikipedia.)

(b) The set $X = \{A \in M_n(\mathbb{R}) : A^t A = I\}$ of orthogonal matrices.

Solution. □

[3] Check this!

[4] A metric space in which every Cauchy sequence converges is called *complete*. This will come up again.