Homework 6

Math 241

Due December 6, 2019 by 5pm

Topics covered: homology of cell complexes, cellular homology, Euler characteristic

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.

Problem 1. *Let X be the union of $S^2 \subset \mathbb{R}^3$ with the line segment connecting the north and south poles. Give X a cell structure and use it to compute $H_*(X)$. (Remark: We know X is homotopy equivalent to $S^2 \vee S^1$, but this fact is antithetical to the exercise.)*

Solution. □

Problem 2 (Hatcher 2.2.15)**.** *Let X be a cell complex. Show that the kernel of the cellular boundary map d_k is isomorphic to $H_k(X^k)$. Conclude that $H_k(X^k)$ is free.*

Solution. □

Problem 3 (Hatcher 2.2.11)**.** *Consider the quotient space X of a cube I^3 obtained by identifying each square face with the opposite square face via a right handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Compute the homology groups of this complex.*

Solution. □

Problem 4 (Hatcher 2.2.21)**.** *Let X be a finite cell complex that is a union of subcomplexes A and B. Show $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$. Prove that if M, N are n-dimensional manifolds, then $\chi(M \# N) = \chi(M) + \chi(N) - \chi(S^n)$.*

Solution. □

Problem 5. *Critique the following argument. "Since the short exact sequences*

$$0 \to S_k(A) \to S_k(X) \to S_k(X, A) \to 0$$

always split, there is always a splitting $H_k(X) \cong H_k(A) \oplus H_k(X, A)$."

Solution. □

Problem 6. *Show that if a closed orientable surface S_g of genus g is a covering space of S_h, then $g = d(h-1)+1$, where d is the degree of the cover. Show for every $d \geq 1$, such a cover exists.*

Solution. □

Problem 7. *Consider the genus-g surface S_g embedded in $\mathbb{R}^3$ in the standard way so that it bounds a region R. Let X be the space obtained from $R \sqcup R$ by identifying the two boundary components by the identity. Compute the reduced homology groups of X using Mayer-Vietoris.*[1]

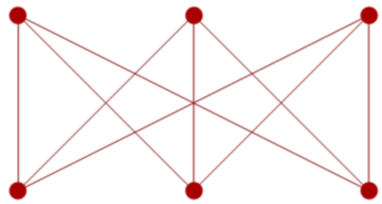
Solution. □

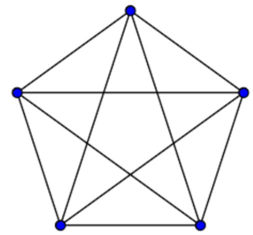
Problem 8 (Hatcher 2.2.24)**.** *Suppose we build S^2 by gluing finitely many polygons along their edges in pairs.*

(a) *Show that the 1-skeleton of the resulting cell structure on S^2 is neither of the graphs $K_{3,3}$ or K_5.* [2]

[1] When $g = 1$, what space do you get? (This doesn't help for doing the computation, but might help with intuition.)

[2] Hint: Use the Euler characteristic. What is the minimum number of edges each face has?





(b) *Deduce that $K_{3,3}$ and K_5 cannot be embedded in $\mathbb{R}^2$. We say that these graphs are not* planar.[3]

Solution. □

[3]Kuratowski's theorem says that a graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a subgraph(!).