Homework 5

Math 241

Due November 22, 2019 by 5pm

Topics covered: axioms for homology, homological algebra, degree theory

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.

Problem 1. *Let ΣX denote the (unreduced) suspension, i.e. the quotient of $X \times I$ by collapsing $X \times \{0\}$ and $X \times \{1\}$ to (separate) points. Show that for any homology theory, there is a natural isomorphism $\tilde{H}_i(X) \to \tilde{H}_{i+1}(\Sigma X)$.*

Solution. □

Problem 2 (Bredon IV.6.3)**.** *Let X, Y be cell complexes. Show that the inclusions of X, Y into $X \vee Y$ induce an isomorphism $\tilde{H}_k(X) \oplus \tilde{H}_k(Y) \cong \tilde{H}_k(X \vee Y)$ whose inverse is induced by the projections of $X \vee Y$ to X and Y. Give a proof that works for any homology theory.*[1]

Solution. □

Problem 3. *Let E_* be a homology theory with coefficient group G. For $X \neq \varnothing$, define reduced homology $\tilde{E}_k(X)$ as the kernel of the map $\epsilon_* : E_k(X) \to E_k(\mathrm{pt})$ induced by $\epsilon : X \to \mathrm{pt}$.*

(a) *Show that there is an exact sequence $0 \to \tilde{E}_k(X) \to E_k(X) \xrightarrow{\epsilon_*} E_k(\mathrm{pt}) \to 0$. Show this sequence splits and conclude that $E_0(X) \cong \tilde{E}_0(X) \oplus G$ and $\tilde{E}_k(X) \cong E_k(X)$ for $k \neq 0$.*

(b) *Fix a pair (X, A) with $A \neq \varnothing$. Show that the long exact sequence in homology induces a long exact sequence in reduced homology*[2]

$$\cdots \to \tilde{E}_k(A) \to \tilde{E}_k(X) \to E_k(X, A) \to \tilde{E}_{k-1}(A) \to \cdots$$

Solution. □

Problem 4. *Let $X = \mathbb{C}^\times$ and $A = \{1, \ldots, n\}$. Compute $H_k(X, A)$ for each k. When $H_k(X, A) \neq 0$, give a generating set.*

Solution. □

Problem 5 (Bredon IV.6.2,4)**.**

(a) *Show that if $f : S^n \to S^n$ is a map without fixed points, then $\deg(f) = (-1)^{n+1}$.*

(b) *Assume n is even. Show that any map $f : \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point.*[3]

Solution. □

Problem 6 (Hatcher 2.1.31)**.**

(a) *Prove the 5-lemma: given a commutative diagram of abelian groups as below, if the rows are exact and α, β, δ, ϵ are isomorphisms, then γ is an isomorphism.*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

[1] Suggestion: consider the long exact sequence of the pair $(X \vee Y, X)$.
[2] Hint: consider the map between long exact sequences induced by the map $(X, A) \to (\mathrm{pt}, \mathrm{pt})$.
[3] Hint: reduce to a statement about $S^n \to S^n$.

(b) *Examine your proof and find minimal assumptions on* $\alpha, \beta, \delta, \epsilon$ *for the same conclusion to hold.*

(c) *Give an example of a diagram where the rows are exact, the maps* $\alpha, \beta, \delta, \epsilon$ *are zero (not necessarily isomorphisms), but* γ *is not zero.*[4]

Solution. □

[4]Hint: you can do this with each group either $\mathbb{Z}$ or 0.