Homework 9

Math 25b

Due April 18, 2018

Topics covered: forms, differential forms, pullbacks, exterior derivatives, chains

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Michele

Problem 1 (Munkres, 27-2)**.** *Write each of the following 9-forms on $\mathbb{R}^{13}$ as an elementary form*

(a) $dx_7 \wedge dx_{10} \wedge dx_6 \wedge dx_9 \wedge dx_3 \wedge dx_{13} \wedge dx_4 \wedge dx_5 \wedge dx_2$

(b) $dx_6 \wedge dx_{11} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_8 \wedge dx_{10} \wedge dx_5 \wedge dx_9$

(c) $dx_{12} \wedge dx_2 \wedge dx_9 \wedge dx_8 \wedge dx_3 \wedge dx_2 \wedge dx_5 \wedge dx_1 \wedge dx_7$

Solution. □

Problem 2. *Let $\omega = dx_1 \wedge \cdots \wedge dx_n \in \Lambda^n(\mathbb{R}^n)$. A basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$ is called positively oriented if $\omega(v_1, \ldots, v_n) > 0$ and is called negatively oriented if $\omega(v_1, \ldots, v_n) < 0$. Determine whether each of the following bases is positively oriented or negatively oriented.*[1]

(a) $4e_1 + 5e_2, 8e_1 - e_2$ *on* $\mathbb{R}^2$

(b) e_3, e_1, e_2 *on* $\mathbb{R}^3$

(c) $-e_2, e_1, -e_3$ *on* $\mathbb{R}^3$

(d) $e_2 + e_1, e_3, e_2, e_4$ *on* $\mathbb{R}^4$

(e) e_3, e_2, e_4, e_1 *on* $\mathbb{R}^4$

Solution. □

Problem 3 (Spivak, CoM 4-6)**.** *Fix vectors $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ and define $\phi \in \Lambda^1(\mathbb{R}^n)$ by*

$$\phi(v) = dx_1 \wedge \cdots \wedge dx_n(v_1, \ldots, v_{n-1}, v).$$

(a) Recall the representation theorem from Math 25a, and use it to deduce there exists a vector $w \in \mathbb{R}^n$ so that

$$\langle v, w \rangle = \det(v_1, \ldots, v_{n-1}, v)$$

for every $v \in \mathbb{R}^n$. This vector w is called the cross product of $v_1, \ldots, v_{n-1}$, and we write $w = v_1 \times \cdots \times v_{n-1}$.

(b) Interpret the cross product on $\mathbb{R}^2$.

(c) Show that if $v_1, \ldots, v_{n-1}$ are linearly independent, then $v_1, \ldots, v_{n-1}, v_1 \times \cdots \times v_{n-1}$ is positively oriented.

Solution. □

[1] For vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ you can tell if a basis is positively or negatively oriented using the *right-hand rule.*

2 For Charlie

Problem 4 (Spivak, CoM 4-9). *Prove the following properties of the cross product in* $\mathbb{R}^3$.

(a) $v \times w = (v_2w_3 - v_3w_2)e_1 + (v_3w_1 - v_1w_3)e_2 + (v_1w_2 - v_2w_1)e_3$. *Hint: recall that the coefficients of a vector* $z \in \mathbb{R}^n$ *with respect to the standard basis/inner product are given by* $z = \sum \langle z, e_i \rangle e_i$.

(b) $|v \times w| = |v| \cdot |w| \cdot |\sin\theta|$ *where* $\theta \in [0, \pi]$ *is the angle between* v *and* w. *Furthermore,*

$$\langle v \times w, v \rangle = \langle v \times w, w \rangle = 0.$$

Hint: Let z *be a unit vector in the orthogonal complement of span*(v, w). *Recall that* $\det(v, w, z)$ *can be interpreted as the volume of a certain parallelepiped. Show that this parallelepiped has volume* $|v| \cdot |w| \cdot \sin\theta$. *How does* $\langle z, v \times w \rangle$ *relate to* $|v \times w|$*?*

(c) *For any vectors* v, w, z,

$\langle v, w \times z \rangle = \langle w, z \times v \rangle = \langle z, v \times w \rangle$

$v \times (w \times z) = \langle v, z \rangle w - \langle v, w \rangle z$

$(v \times w) \times z = \langle v, z \rangle w - \langle w, z \rangle v$

Use the last two to conclude that the cross product is not *associative (choose specific vectors!).*

Hint: to prove the last two use (a). Also, your proof of the third equality should be "the proof in this case is entirely similar to the proof of the second equality".

(d) $|v \times w| = \sqrt{\langle v, v \rangle \cdot \langle w, w \rangle - \langle v, w \rangle^2}$.

Solution. □

Problem 5 (Munkres, 30-4). *Let* $A = \mathbb{R}^2 \setminus \{0\}$. *Consider the 1-form on* A *defined by*

$$\omega = (x\,dx + y\,dy)/(x^2 + y^2).$$

(a) *Show that* $d\omega = 0$. *In this case we say that* ω *is* <u>*closed*</u>.

(b) *Show that in polar coordinates* $\omega = \frac{1}{r}\,dr$. *In other words, consider the polar coordinates transformation* $\phi(r, \theta) = (r\cos\theta, r\sin\theta)$, *and compute* $\phi^*\omega = \frac{1}{r}\,dr$.

(c) *Show that there is a function* f *so that* $df = \omega$. *In this case we say that* ω *is* <u>*exact*</u>. *Hint: Use (b) and the fundamental theorem of calculus. Find* $g(r, \theta)$ *so that* $dg = D_1g\,dr + D_2g\,d\theta$ *is equal to* $\omega = \frac{1}{r}\,dr$. *Then switch back to* x, y *coordinates.*

Solution. □

Problem 6 (Munkres, 27-4; Spivak 4-2). *Let* $f \in \Lambda^k(\mathbb{R}^n)$ *and let* $T : \mathbb{R}^m \to \mathbb{R}^n$ *be a linear map. Define* $T^*f : V \to \mathbb{R}$ *by* $T^*f(v_1, \ldots, v_k) = f(Tv_1, \ldots, Tv_k)$.

(a) *Show that* $T^*f \in \Lambda^k(\mathbb{R}^m)$.

(b) *Assume* $m = n$. *Show that* $T^*(dx_1 \wedge \cdots \wedge dx_n)$ *is* $\det T \cdot dx_1 \wedge \cdots \wedge dx_n$.

Solution. □

3 For Ellen

Problem 7 (Munkres, 30-2). *Consider the differential forms*

$$\omega = xy\,dx + 3\,dy - yz\,dz \quad \text{and} \quad \eta = x\,dx - yz^2\,dy + 2x\,dz$$

on $\mathbb{R}^3$. *Verify by direct computation that* $d(d\omega) = 0$ *and* $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta$.

Solution. □

Problem 8. *Let* ω *be a differential* k*-form on* $\mathbb{R}^n$ *and fix* $a \in \mathbb{R}^n$. *True or false:*

(a) If $\omega(x) = 0$ *for all* x *near* a, *then* $\omega(a) = 0$.

(b) If $\omega(a) = 0$, *then* $d\omega(a) = 0$.

Make sure to explain your answer.

Solution. □

Problem 9 (Munkres, 30-5). *In this problem you will show that the 1-form*

$$\omega = \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy$$

is closed but not exact on $A = \mathbb{R}^2 \setminus \{0\}$. [2]

(a) Show ω *is closed.*

(b) Let B *be the complement* $[0, \infty) \times \{0\}$ *in* $\mathbb{R}^2$ *(i.e. the complement of the positive* x*-axis). Observe that for each* $z = (x, y) \in B$, *there is a unique* $0 < t < 2\pi$ *such that*

$$x = |z| \cdot \cos t \quad \text{and} \quad y = |z| \cdot \sin t.$$

Denote the function $(x, y) \mapsto t$ *by* $\phi : B \to (0, 2\pi)$. *Give a formula for* ϕ *and explain why* ϕ *is* C^1 *(which theorem of ours does it follow from?).*

(c) Show that $\omega = d\phi$ *in* B. *Hint:* $\tan\phi = y/x$ *if* $x \neq 0$ *and* $\cot\phi = x/y$ *if* $y \neq 0$.

(d) Show that if g *is a closed 0-form in* B, *then* g *is constant in* B. *Hint: use the multivariable mean value theorem – make sure the hypotheses apply in the way you use it.*[3]

(e) Show that ω *is not exact in* A. *Hint: If* $\omega = df$ *in* A, *then* $f - \phi$ *is constant in* B. *Evaluate the limit of* $f(1, y)$ *as* y *approaches 0 through positive and negative values.*

Solution. □

[2] From this you can conclude that the disk $D = \{(x, y) : x^2 + y^2 < 1\}$ and the punctured disk $D \setminus \{0\}$ are not diffeomorphic. In other words, differential forms can be used to detect the topology of subsets of $\mathbb{R}^n$!

[3] Note that there are open sets $U \subset \mathbb{R}^2$ and functions $f : U \to \mathbb{R}^2$ so that $Df(u) = 0$ for all $u \in U$ but f is not constant.

4 For Natalia

Problem 10 (Spivak, CoM 4-23). *For $R > 0$ and n an integer, define the singular 1-cube $c_{R,n} : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ by*

$$c_{R,n}(t) = (R\cos 2\pi nt, R\sin 2\pi nt).$$

Fix $0 < R_2 < R_1$ and show that there is a singular 2-cube $c : [0,1]^2 \to \mathbb{R}^2 \setminus \{0\}$ such that

$$c_{R_1,n} - c_{R_2,n} = \partial c.$$

Solution. □

Problem 11 (Spivak, CoM 4-25). *Let c be a C^1 singular k-cube and $p : [0,1]^k \to [0,1]^k$ a injective C^1 function such that $p([0,1]^k) = [0,1]^k$ and $\det p'(x) > 0$ for $x \in [0,1]^k$. If ω is a k-form on $\mathbb{R}^n$, show that*

$$\int_c \omega = \int_{c \circ p} \omega.$$

Solution. □

Problem 12 (Spivak, CoM 4-19). *Fix $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$, and view F as a vector field. Define forms*

$$\omega_F = F_1\, dx + F_2\, dy + F_3\, dz$$

$$\eta_F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

(a) Show that $df = \omega_{grad(f)}$, $d(\omega_F) = \eta_{curl(F)}$, and $d(\eta_F) = div(F)\, dx \wedge dy \wedge dz$.

(b) Use (a) to prove that $curl(grad(f)) = 0$ and $div(curl(F)) = 0$.

Solution. □