Homework 8

Math 25b

Due April 11, 2018

Topics covered: Fubini's theorem, partitions of unity, diffeomorphisms

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Ellen

Problem 0. *One of the problems on this assignment has a part that asks you to show something that's false. You'll need to find it. In your solution you should explain why it's false. Good luck!*

Problem 1 (Munkres, 12-2)**.** *Let* $Q = [0,1] \times [0,1]$*. Define* $f : Q \to \mathbb{R}$ *by*

$$f(x,y) = \begin{cases} 1/q & y \in \mathbb{Q} \text{ and } x = p/q \text{ lowest terms} \\ 0 & \text{else} \end{cases}$$

(a) Does $\int_Q f$ *exist? Explain.*

(b) Compute $\underline{\int}_{y \in I} f(x,y)$ *and* $\overline{\int}_{y \in I} f(x,y)$*.*

(c) Verify Fubini's theorem.

Solution. □

Problem 2 (Spivak, CoM 3-28 and Munkres, 12-4 and Hubbard, 4.5.11)**.** *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$*.*

(a) Use Fubini's theorem to give an easy proof that $D_1 D_2 f = D_2 D_1 f$ *if these are continuous (Clairaut's theorem).*

(b) The function

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is the standard example of a function that is twice-differentiable but $D_1 D_2(f) \neq D_2 D_1(f)$ *at 0 (you showed this in HW4). Where does the proof of (a) fail in this case?*[1]

Solution. □

Problem 3 (Spivak, CoM 3-26)**.** *Let* $f : [a,b] \to \mathbb{R}$ *be bounded, integrable, and non-negative. Let* $A = \{(x,y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$*. Show that* A *is rectifiable and has area* $\int_a^b f$*. Hint: most of the work goes toward showing that* A *is rectifiable. Warning:* f *is not assumed to be continuous!*

Solution. □

[1] You might enjoy computing the partial derivatives with Mathematica or Wolfram Alpha.

2 For Charlie

Problem 4 (Spivak, CoM 3-29)**.**

(a) *Use Fubini's theorem to derive the volume of a cone C with base r and height h.*

(b) *Fix $a \geq 0$, and let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(z) \leq g(z)$ for each $z \in [a, b]$. Consider $S = \{(y, z) : f(z) \leq y \leq g(z) \text{ and } a \leq z \leq b\}$. Derive an expression for the volume of a set $C \subset \mathbb{R}^3$ obtained by revolving S about the z-axis.*

(c) *Repeat (b) but now with f, g functions of y, i.e. $S = \{(y, z) : a \leq y \leq b \text{ and } f(y) \leq z \leq g(y)\}$. (Again revolving S around the z-axis.)* [2]

Solution. □

Problem 5 (Spivak, CoM 3-30)**.** *Let C be the set constructed in HW3#3. Show that*

$$\int_{y\in[0,1]} \left(\int_{x\in[0,1]} \chi_C(x, y) \right) = \int_{x\in[0,1]} \left(\int_{y\in[0,1]} \chi_C(x, y) \right) = 0$$

but that $\int_{[0,1]\times[0,1]} \chi_C$ does not exist.

Solution. □

Problem 6 (Spivak, CoM 3-36)**.** *In this problem you prove Cavalieri's principle.*

(a) *Let A and B be rectifiable subsets of $\mathbb{R}^3$. Let $A_c = \{(x, y) : (x, y, c) \in A\}$ and define B_c similarly. Suppose A_c and B_c are rectifiable and have the same area for each c. Show that A and B have the same volume.*[3]

(b) *Look up the "napkin-ring problem," which is a popular application of Cavalieri's principle. Explain it to your friends.*

Solution. □

[2] In multivariable calculus, these two methods of computing volumes of revolution are typically called the "shell" and the "washer" methods.

[3] See pictures on the course webpage.

3 For Natalia

Problem 7 (Munkres, 16-1). *In this problem you will show*

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

is smooth $f : \mathbb{R} \to \mathbb{R}$. [4]

(a) *Show that* $x < e^x$ *for all* $x \in \mathbb{R}$. *Hint: use the power series definition.*

(b) *Prove that* f *is continuous at 0.*

(c) *Prove that* f *is differentiable at 0 and* $f'(0) = 0$. *Hint: L'Hopital. It might help to write* $\frac{e^{-1/x}}{x} = \frac{1/x}{e^{1/x}}$.

(d) *For each* $k \geq 1$ *the functions* $f^{(k)}(x)$ *are linear combinations of the functions* $\frac{1}{x^n}e^{-1/x}$ *on* $(0, \infty)$. *Conclude that* f *is smooth on* $(0, \infty)$.

(e) *Show that* $\lim_{x \to 0+} \frac{1}{x^n}e^{-1/x} = 0$ *for every* $n \geq 1$. *Conclude that* $f^{(k)}(0) = 0$ *for every* k.

Solution. □

Problem 8 (Spivak 2-26). *Let* $h(x) = f(x)f(1-x)$, *where* f *is the function from the previous problem. Observe that* $h : \mathbb{R} \to \mathbb{R}$ *is smooth and that* h *is positive on* $(0, 1)$ *and 0 elsewhere.*

(a) *Show that there is a smooth function* $g : \mathbb{R} \to [0, 1]$ *such that* $g(x) = 0$ *for* $x \leq 0$ *and* $g(x) = 1$ *for* $x \geq \epsilon$. *Hint: if* ϕ *is a smooth function that is positive on* $(0, \epsilon)$ *and 0 otherwise, consider* $g(x) = \int_0^x \phi / \int_0^\epsilon \phi$.

(b) *If* $a \in \mathbb{R}^n$, *define* $\phi : \mathbb{R}^n \to \mathbb{R}$ *by*

$$\phi(x) = h(\frac{x_1 - a_1}{\epsilon}) \cdot \ldots \cdot h(\frac{x_n - a_n}{\epsilon}).$$

Show that g *is smooth, positive on* $Q = (a_1, a_1 + \epsilon) \times \cdots \times (a_n, a_n + \epsilon)$, *and zero elsewhere.*

(c) *If* $A \subset \mathbb{R}^n$ *is open and* $C \subset A$ *is compact, show that there is a non-negative smooth function* $\phi : A \to \mathbb{R}$ *such that* $\phi(x) > 0$ *for* $x \in C$ *and* $\phi = 0$ *outside of some closed set contained in* A.

(d) *Show that we can choose such an* ϕ *so that* $\phi : A \to [0, 1]$ *and* $\phi(x) = 1$ *for* $x \in C$. *Hint: Compose the function from (c) by the function from (a) with a smart choice of* ϵ.

Solution. □

[4] We haven't given a formal treatment to exponential functions, although you probably know some basic properties about them. One rigorous definition is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. (One can show that this series converges for every x.) From this definition one can deduce some familiar properties like $e^0 = 1$ and $e^x \cdot e^y = e^{x+y}$. A different characterization/definition of e^x is as the unique solution of the differential equation $f' = f$ with initial condition $f(0) = 1$ (c.f. Extra Credit 2).

4 For Michele

Problem 9 (Munkres, 16-3)**.** *Fix any* $S \subset \mathbb{R}^n$ *and fix* $y \in S$*. Say that a function* $f : S \to \mathbb{R}$ *is continuously differentiable at* y *if there is a* C^1 *function* $g : U \to \mathbb{R}$ *defined in a neighborhood of* y *in* $\mathbb{R}^n$ *such that* g *agrees with* f *on* $U \cap S$*.*

(a) Suppose $f : S \to \mathbb{R}$ *is continuously differentiable at* y*. Show that if* $\phi : \mathbb{R}^n \to \mathbb{R}$ *is a* C^1 *function whose support lies in* U*, then the function*

$$h(x) = \begin{cases} \phi(x)g(x) & x \in U \\ 0 & x \notin supp(\phi) \end{cases}$$

is a well-defined C^1 *function on* $\mathbb{R}^n$*.*

(b) Prove: If $f : S \to \mathbb{R}$ *is continuously differentiable at each* $y \in S$*, then* f *may be extended to a* C^1 *function* $h : A \to \mathbb{R}$ *defined on an open set containing* S*. Hint: this is a gluing problem.*

Solution. □

Problem 10 (Spivak, CoM 3-40)**.** *Fix* $g : \mathbb{R}^n \to \mathbb{R}^n$*, and suppose* $a \in \mathbb{R}^n$ *satisfies* $\det Dg(a) \neq 0$*.*

(a) Prove that in some open set containing a *we can write* $g = T \circ f_n \circ \cdots \circ f_1$*, where* f_i *is of the form* $f_i(x) = (x_1, \ldots, \phi_i(x), \ldots, x_n)$*, and* T *is a linear map. Hint: Use the map* T *to replace* g *by a function whose derivative at* a *is the identity.*

(b) Show that we can write $g = g_n \circ \cdots \circ g_1$ *if and only if* $Dg(a)$ *is a diagonal matrix.*

Solution. □