Homework 7

Math 25b

Due March, 28 2018

Topics covered: the integral, integrability, measure 0

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Ellen

Problem 1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $K \subset \mathbb{R}^n$ is compact. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ so that for every $x, y \in K$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. This property is called* uniform continuity. *Hint: Use continuity to construct an open cover by balls. It might be helpful to shrink the radius of the balls before taking a finite subcover (compare to the proof of HW3#5).*[1]

Solution. □

Problem 2 (Munkres 10-2)**.** *Let $Q \subset \mathbb{R}^n$ be a closed rectangle. Give a direct proof that if $f : Q \to \mathbb{R}$ is continuous then f is integrable, i.e. show that if f is continuous, then for every $\epsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$. Hint: use uniform continuity.*

Solution. □

Problem 3 (Spivak, CoM 3-3)**.** *Fix a closed rectangle $A \subset \mathbb{R}^n$. In this exercise, you show that the set of integrable functions $f : A \to \mathbb{R}$ has the structure of a vector space. Let $f, g : A \to \mathbb{R}$ be integrable.*

(a) *For any partition P of A and subrectangle S, show that*[2]

$$m_S(f) + m_S(g) \leq m_S(f + g) \text{ and } M_S(f + g) \leq M_S(f) + M_S(g)$$

and therefore

$$L(f, P) + L(g, P) \leq L(f + g, P) \text{ and } U(f + g, P) \leq U(f, P) + U(g, P).$$

Hint: To show $m_S(f)+m_S(g) \leq m_S(f+g)$ is suffices to show $m_S(f)+m_S(g) \leq m_S(f+g)+\epsilon$ for every $\epsilon > 0$.

(b) *Show that $f + g$ is integrable and $\int_A f + g = \int_A f + \int_A g$.*

(c) *For any constant c, show that $\int_A cf = c \int_A f$.*

Solution. □

[1]You could have solved this problem back when we learned about compactness. This exercise appears here because it could be useful for other problems below. Also I don't want you to forget about compactness! Note also that Munkres' proves this in Theorem 4.7. However, in a way, his argument is more complicated than necessary. I want you to give a different proof.

[2]Here $m_S(f), M_S(f)$ is shorthand for our notation $\min_S(f), \max_S(f)$ from class. Both Munkres and Spivak use this abbreviated notation.

2 For Natalia

Problem 4 (Spivak, CoM 3-1). *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by*

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Show that f is integrable and $\int_{I^2} f = \frac{1}{2}$.

Solution. □

Problem 5 (Spivak, CoM 3-6). *Fix a closed rectangle $A \subset \mathbb{R}^n$. If $f : A \to \mathbb{R}$ is integrable, show that $|f|$ is integrable and $|\int_A f| \leq \int_A |f|$.*

Solution. □

Problem 6 (Spivak, CoM 3-4). *Fix a closed rectangle $A \subset \mathbb{R}^n$. Let $f : A \to \mathbb{R}$ and let P be a partition of A. Show that f is integrable if and only if for each subrectangle S (of P) the restriction $f|_S$ is integrable, and that in this case* [3]

$$\int_A f = \sum_S \int_S f|_S.$$

Solution. □

[3] This is an example of a "local-to-global" principle: the integral over A can be computed by computing the integral "locally" over subrectangles of A. This will appear again when we extend our definition of the integral to allow A to be an open set.

3 For Michele

Problem 7. *Let $Q \subset \mathbb{R}^n$ be a closed rectangle, and assume $f : Q \to \mathbb{R}$ is integrable.*

(a) Show that if $m \leq f(x) \leq M$ for all $x \in Q$, then $m \cdot v(Q) \leq \int_Q f \leq M \cdot v(Q)$.

(b) Show that if $Q = [a, b]$, then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_{[a,x]} f$$

is continuous (even if f is not continuous!). Hint: You'll want to use Problem 6.

Solution. □

Problem 8 (Munkres 11-6)**.** *Let $f : [a, b] \to \mathbb{R}$. Show that the graph $G_f = \{(x, f(x)) : x \in [a, b]\}$ has measure 0 in $\mathbb{R}^2$. Hint: use uniform continuity.*

Solution. □

Problem 9 (Spivak, CoM 3-14)**.** *Fix a closed rectangle $A \subset \mathbb{R}^n$. Show that if $f, g : A \to \mathbb{R}$ are integrable, so is $f \cdot g$. Hint: don't prove this using the definition.*

Solution. □

4 For Charlie

Problem 10. *Read Munkres §2, especially Theorem 2.4. It covers some linear-algebra facts that we'll need soon.*

(a) *Let* $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Give elementary matrices* $E_1, \ldots, E_k$ *of the form*

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{1}$$

so that $E_k \cdots E_1 A = I$. *(This problem asks you to row-reduce the matrix A without using the "swap two rows" move.)*

(b) *Write each of the linear maps (1) in coordinates as a function* $g(x, y) = (g_1(x, y), g_2(x, y))$ *and observe that each function either fixes the* x *coordinate or fixes the* y *coordinate.*

Solution. □

Problem 11. *Let* $A \subset \mathbb{R}^n$ *be open and let* $g : A \to \mathbb{R}^n$ *be* C^1. *Prove or give a counterexample: If* $E \subset A$ *has content 0, then* $g(E)$ *has content 0. Hint: Note the relation with Munkres Lemma 18.1.*

Solution. □

Problem 12 (Spivak 3-10)**.** (a) *Does the Cantor set*[4] *have measure 0? content 0?*

(b) *Give an example of a set* X *for which* X *has measure 0 but* $bd(X)$ *does not have measure 0. Hint: A certain exercise from a previous homework assignment might be helpful.*

Solution. □

[4] I can't remember if we've seen this. This set is constructed in a way similar to the Sierpinski carpet. See Wikipedia.