Homework 6

Math 25b

Due March, 21 2018

Topics covered: implicit function theorem, manifolds, Lagrange multipliers

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds.* I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Natalia

Problem 1. *Let* $A \in M_{m\times n}(\mathbb{R})$ *be an* $m \times n$ *matrix, which we also view as a linear map* $\mathbb{R}^n \to \mathbb{R}^m$*. Suppose that* A *has rank* k*, i.e.* $\dim \operatorname{Im} A = k$*. Show that* A *has a* $(k \times k)$ *sub-matrix (obtained by selecting* k *rows and* k *columns) that is invertible. Hint: You're going to need some linear algebra facts from last semester. If you want more suggestions, ask me.*

Solution. □

Problem 2 (Spivak, CoM 5-5)**.** *Prove that a* d*-dimensional vector subspace* $W \subset \mathbb{R}^n$ *is a* d*-dimensional manifold directly using the definition of a manifold.*

Solution. □

Problem 3 (Munkres, 9-5)**.** *Let* $f, g : \mathbb{R}^3 \to \mathbb{R}$ *be* C^1 *functions. In good situation (to be made precise below), one expects that each of the equations* $f(x, y, z) = 0$ *and* $g(x, y, z) = 0$ *represents a smooth surface in* $\mathbb{R}^3$ *and that their intersection is a smooth curve.*[1] *Show that if* $p = (a, b, c)$ *satisfies both equations and if* $\partial(f, g)/\partial(x, y, z)(d)$ *has rank 2, then near* d*, one can solve these equations for two of* x, y, z *in terms of the third, thus representing the solution set locally as a parameterized curve. Hint: implicit function theorem.*

Solution. □

[1] In the simplest case, $f = 0$ and $g = 0$ are two planes in $\mathbb{R}^3$ intersecting in a line.

2 For Ellen

Problem 4 (Munkres, 9-6)**.** *Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be C^1. Suppose $f(a) = 0$ and $Df(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution. Hint: it might help to revisit the proof of the implicit function theorem or the manifold recognition theorem.*

Solution. □

Problem 5 (Hubbard, 3.1.1)**.** *For what values of c is the set $Z_c = \{(x, y) : \sin(x + y) = c\}$ a smooth curve? What is the equation for the tangent line to such a curve at a point (u, v).*

Solution. □

Problem 6. *(a) For what values of a and b are the sets $X_a = \{(x, y, z) : x - y^2 = a\}$ and $Y_b = \{(x, y, z) : x^2 + y^2 + z^2 = b\}$ 2-manifolds in $\mathbb{R}^3$?*

(b) For what values of a and b is the intersection $X_a \cap Y_b$ a 1-manifold? What geometric relation is there between X_a and Y_b for other values of a and b?

Solution. □

3 For Michele

Problem 7 (Hubbard, 3.1.16). *Show that the set $X \subset \mathbb{R}^3$ given by $x^3 + xy^2 + yz^2 + z^3 = 4$ is a smooth surface. What is the equation of the tangent plane to X at $(1, 1, 1)$?*

Solution. □

Problem 8 (Hubbard, 3.2.1). *Consider the space X_ℓ of positions of a rod of length ℓ in $\mathbb{R}^3$, where one endpoint is constrained to the x-axis and the other is constrained to be on the unit sphere centered at the origin.*

(a) Give equations for X_ℓ as a subset of $\mathbb{R}^4$, where the coordinates are the endpoint of the rod on the x-axis (t say) and the coordinates (x, y, z) of the other end of the rod.

(b) Show that near the point $(1 + \ell, 1, 0, 0)$ the set X_ℓ is a manifold and give the equation of its tangent space.

(c) Show that for $\ell \neq 1$, X_ℓ is a manifold.

Solution. □

Problem 9 (Hubbard, 3.2.2). *Consider the space X of positions of a rod of length 2 in $\mathbb{R}^3$ where one point is constrained to the sphere with equation $(x - 1)^2 + y^2 + z^2 = 1$ and the other on the sphere of equation $(x + 1)^2 + y^2 + z^2 = 1$.*

(a) Give equations for X as a subset of $\mathbb{R}^6$. Use coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) for the endpoints of the rod on the first and second sphere, respectively.

(b) Show that near the point $(1, 1, 0, -1, 1, 0)$ the set X is a manifold and give the equation of its tangent space. What is the dimension of X near this point?

(c) Find the two points of X near which X is not a manifold. Give an informal explanation.[2]

Solution. □

[2]Note: the manifold recognition/implicit function theorem is *not* an if and only if, so you can't use this to prove that X is not a manifold at a point.

4 For Charlie

Problem 10 (Hubbard 3.7.7). *Find the minimum of the function* $\phi(x, y, z) = x^3 + y^3 + z^3$ *on the intersection of the planes* $x + y + z = 2$ *and* $x + y - z = 3$.

Solution. ☐

Problem 11 (Hubbard 3.7.9). *What is the volume of the largest rectangular parallelepiped contained in the ellipsoid* $x^2 + 4y^2 + 9z^2 \leq 9$*?*

Solution. ☐

Problem 12 (Hubbard 3.7.15). *(a) Show that the set* $X \subset M_2(\mathbb{R})$ *of matrices with determinant* 1 *is a manifold. What is its dimension?*

(b) Find a matrix in X *that is closest to the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Hint: avoid minimizing a function with square roots.*

Solution. ☐