Homework 3

Math 25b

Due February, 14 2018

Topics covered: subsets of $\mathbb{R}^n$, compactness, Heine–Borel, derivatives

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Natalia

Problem 1 (Spivak, C 8-15). *Use the Onion Ring Theorem to give an alternate proof of the following statement of the Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there exists $c \in [a, b]$ with $f(c) = 0$. Hint: You can do this with the "method of bisection" similar to our proof that closed rectangles are compact. If you are stuck, see the problem statement in Spivak for more instruction.*

Solution. □

Problem 2 (Spivak, CoM 1-14).

(a) Prove that the union of any collection of open sets is open. Prove the intersection of two (and hence countably many) open sets is open. Conclude that the the intersection of any number of closed sets is closed.

(b) Use (a) to explain why the Sierpinski triangle $\mathcal{S} \subset \mathbb{R}^2$ is compact.[1] *(To do this completely rigorously would be tedious. Here I'd like you to give an explanation in words.)*

Solution. □

Problem 3 (Spivak, CoM 1-17). *Construct a set $A \subset [0, 1] \times [0, 1]$ such that A contains at most one point on each horizontal and each vertical line but the boundary of A is $[0, 1] \times [0, 1]$. (Hint: it suffices to ensure that A contains points in each quarter of the square, and each sixteenth, etc.)*

Solution. □

[1] You may want to read more about the construction of the Sierpinski triangle. See e.g. Wikipedia.

2 For Michele

Problem 4 (Spivak, CoM 1-20). *In class we stated the Heine–Borel theorem: $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*[2] *In this problem you learn how to prove it. Start by reading Corollary 1-7 (in CoM), which proves one direction. Next prove the converse: A compact subset of $\mathbb{R}^n$ is closed and bounded.*

Solution. □

Problem 5 (Spivak, CoM 1-21). *(a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.*

(b) If A is closed and B is compact and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. (Hint: For each $b \in B$ find an open set U containing b such that this relation hods for $x \in U \cap B$.)

(c) Give a counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact.

Solution. □

Problem 6 (Spivak, CoM 1-22). *If U is open and $C \subset U$ is compact, show that there is a compact set D such that $C \subset \operatorname{int}(D)$ and $D \subset U$.*

Solution. □

[2] Spivak does not call this Heine–Borel, but he is atypical in this sense. See e.g. the Wiki article on Heine–Borel. Spivak calls the theorem that closed rectangles are compact Heine–Borel. This might be justified since showing closed rectangles are compact is the "hard part" and the characterization "compact if and only if closed and bounded" is an easier corollary.

3 For Ellen

Problem 7 (Spivak, CoM 1-29). *(a) Prove that if $A \subset \mathbb{R}^n$ is compact and $f : A \to \mathbb{R}^m$ is continuous, then the image $f(A) = \{f(a) : a \in A\}$ is compact.*

(b) If A is compact, prove that every continuous function $f : A \to \mathbb{R}$ takes on a maximal value.

Solution. □

Problem 8 (Spivak, C 9-18). *Let $f : [0,1] \to \mathbb{R}$ be Thomae's function. Prove that f is not differentiable at any $a \in (0,1)$. Hint: there's an easy argument that f is not differentiable at each rational number.*

Solution. □

Problem 9 (Spivak, C 9-21, 9-28). *(a) Explain why derivatives are a "local property": if $f(x) = g(x)$ for all x in some open interval containing a, then $f'(a) = g'(a)$.*

(b) Find $f'(x)$ and $f''(x)$ for $f(x) = |x|^3$. Does $f'''(x)$ exist for all x?

Solution. □

4 For Charlie

Problem 10 (Spivak, CoM 2-32). *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Compute f' where it's defined. Show that f' is not continuous at 0. [3]

Solution. □

Problem 11 (Spivak, C 10-27). *Suppose that f is differentiable at 0 and $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g that is continuous at 0.*

Solution. □

Problem 12 (Spivak, C 10-2). *Find $f'(x)$ for each of the following functions f.*[4]

(i) $f(x) = \sin((x+1)^2(x+2))$

(ii) $f(x) = \sin^3(x^2 + \sin x)$

(iii) $f(x) = \sin^2((x + \sin x)^2)$

(iv) $f(x) = \sin((\sin^7 x^7 + 1)^7)$

Solution. □

[3] If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous, we call f continuously differentiable. In this problem you show that not all differentiable functions are continuously differentiable.

[4] According to Spivak, "Although rapid calculation is not the goal of mathematics, if you hope to treat theoretical applications of the Chain Rule with aplomb, these concrete applications should be child's play – mathematicians like to pretend that they can't even add, but most of them can when they have to."