Homework 2

Math 25b

Due February, 7 2018

Topics covered: continuity, continuity theorems, least upper bound property, metric spaces

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds.* I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).

1 For Ellen

Problem 1 (Spivak, CoM 1-5)**.**

(a) *Show that for* $x, y \in \mathbb{R}^n$*, one has*

$$\big||x| - |y|\big| \leq |x - y|.$$

This is called the reverse triangle inequality. Hint: use the triangle inequality.

(b) *Conclude that if* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous, then* $|f| : \mathbb{R}^n \to \mathbb{R}$ *defined by* $|f|(x) = |f(x)|$ *is continuous.*

Solution. □

Problem 2 (Spivak, CoM 1-10 and 1-25)**.** *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map.*

(a) *Show that there is a number* M *such that* $|Th| \leq M|h|$ *for* $h \in \mathbb{R}^n$*. Hint: Estimate* $|Th|$ *in terms of* $|h|$ *and the entries in the matrix of* T*.*

(b) *Show that* T *is continuous.*

Solution. □

Problem 3 (Spivak, C 7-10 and 7-11)**.**

(a) *Suppose* f, g *are continuous on* $[a, b]$ *and that* $f(a) < g(a)$*, but* $f(b) > g(b)$*. Prove that* $f(c) = g(c)$ *for some* $c \in [a, b]$*.*

(b) *Suppose* f *is continuous on* $[0, 1]$ *and* $0 \leq f(x) \leq 1$ *for all* $x \in [0, 1]$*. Prove that* $f(x) = x$ *for some number* x*. This is a special case of the* Brouwer fixed point theorem*.*

Hint: if your proofs are not very short, then they are not the right ones.

Solution. □

2 For Natalia

Problem 4 (Spivak, C 7-13). *Consider* $f : [-\frac{2}{\pi}, \frac{2}{\pi}] \to \mathbb{R}$ *defined by* $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$

(a) *Show that* f *satisfies the conclusion of the Intermediate Value Theorem (i.e. for any* d *with* $f(a) < d < f(b)$*, there exists* $c \in [a, b]$ *with* $f(c) = d$*). Does* f *satisfy the hypothesis of IVT?*

(b) *Suppose that* $f : [a, b] \to \mathbb{R}$ *satisfies the conclusion of IVT and that* f *takes on each value only once (i.e.* f *is injective). Prove that* f *is continuous. Hint: It might help to show that* f *is either increasing or decreasing (recall that* f *is increasing if* $x < y$ *implies* $f(x) < f(y)$*).*

Solution. □

Problem 5 (Spivak, C 7-20). *Prove that there does not exist a continuous function on* $\mathbb{R}$ *that takes on every value exactly twice. Is the same statement true if we replace twice by thrice?*

Solution. □

Problem 6 (Spivak, C 7-16). *Let* $p(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial* $p \in \mathrm{Poly}(\mathbb{R})$*. Prove that there is a number* y *so that* $|p(y)| \leq |p(x)|$ *for all* $x \in \mathbb{R}$*. Hint: if you're stuck, it may help to read the proof of Theorem 10 in Chapter 7 of Spivak's Calculus.*

Solution. □

3 For Charlie

Problem 7 (Spivak, C 8-5)**.** *In this problem,* $x, y \in \mathbb{R}$. [1]

(a) *Suppose* $y - x > 1$. *Prove that there is an integer* k *such that* $x < k < y$. *Hint: let* ℓ *be the smallest integer less than* x *and consider* $\ell + 1$.

(b) *Suppose* $x < y$. *Prove that there is a rational number* r *such that* $x < r < y$. *Hint: Find* $n \in \mathbb{N}$ *so that* $n(y - x) > 1$.

(c) *Suppose* $r < s$ *are rational numbers. Prove that there is an irrational number between* r *and* s. *Hint: You may find it useful that* $\sqrt{2}$ *is irrational.*

(d) *Suppose* $x < y$. *Prove that there is an irrational number between* x *and* y.

A subset $A \subset \mathbb{R}$ *is called* <u>*dense*</u> *if every open interval contains an element of* A. *The problem above shows that the rational numbers* $\mathbb{Q}$ *and the irrational numbers* $\mathbb{R} \setminus \mathbb{Q}$ *are both dense in* $\mathbb{R}$.

Solution. □

Problem 8 (Spivak, C 8-6 and 8-7)**.**

(a) *Show that if* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $f = 0$ *for all* x *in a dense set* A, *then* $f(x) = 0$ *for all* x. [2]

(b) *Use (a) to prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $f(x + y) = f(x) + f(y)$ *for all* x, y *then* f *is linear, i.e. there exists* c *so that* $f(x) = cx$ *for all* x. [3]

Solution. □

Problem 9 (Spivak, C 8-8)**.** *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is such that* $f(y) \leq f(z)$ *whenever* $y < z$ *(such a function is sometimes called nondecreasing). Prove that* $\lim_{x \to a-} f(x)$ *and* $\lim_{x \to a+} f(x)$ *both exist. Hint: first decide what each limit should equal.*

Solution. □

[1] For this problem, at several points, you might think a certain statement is "obvious," but it's useful to remember that there are ordered fields that don't satisfy the LUB property, where the set of natural numbers is bounded(!) and where $0 \neq \inf\{\frac{1}{n} : n \in \mathbb{N}\}$ (!).

[2] Compare with Thomae's function.

[3] There exist functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x + y) = f(x) + f(y)$ for all x, y, but are not continuous, but they are not easy to construct.

4 For Michele

Definition. Let X be a set. A metric on X is a function $d : X \times X \to \mathbb{R}$ that satisfies

(i) (positivity) $d \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) (symmetry) $d(x, y) = d(y, x)$ for each $x, y \in X$, and

(iii) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

The pair (X, d) is called a metric space. Given a metric space (X, d), a point $x \in X$, and $r > 0$, we denote the *open d-ball around x of radius r*

$$B_r(x, d) = \{x \in X : d(y, x) < r\}.$$

Last quarter, we showed that if $X = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is the standard (or any) inner product, then

$$d(x, y) := ||x - y|| = \langle x - y, x - y \rangle^{1/2} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

is a metric on X. We will call this the norm metric on $\mathbb{R}^n$.

There are many other metrics on $\mathbb{R}^n$. Another interesting metric is the sup metric. It is defined as

$$d'(x, y) = \sup_{i=1,\ldots,n} |x_i - y_i|.$$

In other words, $d'(x, y)$ is the maximum value of the finite set $\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$. You should convince yourself that d' is a metric on $\mathbb{R}^n$. Note that the sets $B_r(x, d')$ are open rectangles! The metric d' is sometimes also called the *taxi-cab metric* (why does this make sense?). With this naming convention, it would make sense to call the metric d the *as-the-crow-flies metric.*

Problem 10. *Fix $n \geq 1$. Consider $\mathbb{R}^n$ with the norm metric d and the sup metric δ. Show that there are constants $0 < K, L$ so that $K \cdot d'(x, y) \leq d(x, y) \leq L \cdot d'(x, y)$ for all $x, y \in \mathbb{R}^n$. Use this to conclude that every d-metric ball around x contains a d'-metric ball around x and vice versa.*

Solution. □

Problem 11. *Prove the following are equivalent:* [4]

(i) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

(ii) For every open set U, the preimage $f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}$ is open.

Relevant side question: How do you re-phrase the definition of continuity in terms of metric balls? Remark: This problem is similar to but slightly different from Theorem 1-8 in Spivak's Calculus on Manifolds.

Solution. □

[4] Since these statements are equivalent, we could have defined continuity with the second definition. The benefit to doing this is that definition (ii) can be made for topological spaces, a class of spaces more general than metric spaces. There are other benefits to definition (ii), as we will eventually see.