Homework 1

Math 25b

Due January 31, 2018

Topics covered: In this problem set you will get familiar with working with limits. Several of the exercises ask you to prove basic properties that we will use over and over throughout the course.

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).

1 For Charlie

Problem 1 (Spivak, C 5-9). *Suppose* $\lim_{x\to a} f(x) = L$. *Show that* $\lim_{h\to 0} f(a+h) = L$.

Solution. ☐

Problem 2 (Spivak, C 5-18). *(a) Prove that if* $\lim_{x\to a} f(x) = L$, *then* f *is bounded near* a, *i.e. there exists* M *so that* $|f(x)| < M$ *if* $0 < |x-a| < \delta$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ *is continuous at* a *and* $f(a) > 0$, *then there exists* $\delta > 0$ *so that* $f(x) > 0$ *for all* $x \in (a-\delta, a+\delta)$.

Solution. ☐

Problem 3 (Spivak, C 5-21). *Show that if* g *is bounded and* $\lim_{x\to a} f = 0$, *then* $\lim_{x\to a} f(x)g(x) = 0$.

Solution. ☐

2 For Ellen

Problem 4 (Spivak, C 5-29)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function. We say that* f *approaches* b *as* x *approaches* a *from the right if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that* $|f(x) - b| < \epsilon$ *whenever* $0 < x - a < \delta$*. In this case we write* $\lim_{x \to a+} f(x) = b$*. Similarly, we say that* f *approaches* b *as* x *approaches* a *from the left if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that* $|f(x) - b| < \epsilon$ *whenever* $0 < a - x < \delta$*. Then we write* $\lim_{x \to a-} f(x) = b$*. Show that the following statements are equivalent.*

(i) $\lim_{x \to a} f = b$

(ii) $\lim_{x \to a+} f = b$ *and* $\lim_{x \to a-} f = b$.

Solution. □

Problem 5 (Spivak, C 5-2)**.** *Find the following limits. Hint: it helps to start with an algebraic simplification.*

(a) $\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x}$

(b) $\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x^2}$

Solution. □

Problem 6 (Spivak, C 5-4)**.** *Decide for which numbers* a *the limit* $\lim_{x \to a} f(x)$ *exists. You don't have to prove anything. (Suggestion: graph each of the functions.)*

(a) $f(x) = x - [x]$. *(Remark: If you want to graph the sum of two functions, it might help to first graph each of the functions individually.)*

(b) $f(x) = [x] + \sqrt{x - [x]}$.

(c) $f(x) = [1/x]$.

(d) $f(x) = 1/[1/x]$.

Here $[x]$ *denotes the greatest integer less than* x*, e.g.* $[4.3] = [4] = 4$ *and* $[-4.3] = -5$.

Solution. □

3 For Natalia

Problem 7 (Spivak, C 5-3)**.** *In each case, find* δ *such that* $|f(x) - \ell| < \epsilon$ *for all* x *satisfying* $0 < |x - a| < \delta$.

(a) $f(x) = x^4$; $a > 0$, $\ell = a^4$.

(b) $f(x) = \frac{1}{x}$; $a = 1$, $\ell = 1$.

(c) $f(x) = x^4 + \frac{1}{x}$; $a = 1$, $\ell = 2$.

Solution. $\square$

Problem 8 (Spivak, C 5-12)**.** *Fix* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g : \mathbb{R}^n \to \mathbb{R}$. *Suppose* $\lim_{x\to a} f(x) = K$ *and* $\lim_{x\to a} g(x) = L$.

(a) *Show that if* $f(x) \le g(x)$ *for all* x, *then* $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. *Hint: To show* $K - L \le 0$ *it suffices to show that* $K - L < c$ *for every* $c > 0$.

(b) *How can the hypothesis be weakened?*

(c) *If* $f(x) < g(x)$ *for all* x *does it necessarily follow that* $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$*?*

Solution. $\square$

Problem 9 (Spivak, C 5-13)**.** *In this problem you prove the "Squeeze Theorem." Suppose that* $f(x) \le g(x) \le h(x)$ *and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. *Prove that* $\lim_{x\to a} g(x)$ *exists and*

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

If you're stuck, it may help to draw a picture.

Solution. $\square$

4 For Michele

Problem 10 (Spivak, CoM 1-23). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function with coordinate functions $f = (f_1, \ldots, f_n)$. Show that $\lim_{x\to a} f(x) = b$ if and only if $\lim_{x\to a} f_i(x) \to b_i$ for each $i = 1, \ldots, n$.*

Solution. □

Problem 11 (Spivak, C 5-8). *(a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a} f(x)+g(x)$ or $\lim_{x\to a} f(x)g(x)$ exist?*

(b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) + g(x)$ exists, must $\lim_{x\to a} g(x)$ exist?

(c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a} f(x) + g(x)$ exist?

(d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Solution. □

Problem 12 (Spivak, C 5-17). *Show that $\lim_{x\to 0} 1/x$ does not exist, i.e. show that $\lim_{x\to 0} 1/x = L$ is false for every number L. Hint: It's vitally important to start by writing down the correct negation of the statement "For every $\epsilon > 0$ there exists $\delta > 0$ such that if x satisfies $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$."*

Solution. □