Homework 9

Math 25a

Due November 30, 2018

Topics covered (lectures 18-19): inner products, orthogonality and Gram-Schmidt

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Laura Z.

Problem 1 (Axler 6.A.12). *(a) Use Cauchy–Schwarz to prove that* $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$ *for any real numbers* a, b, c, d*. (You could also solve this by expanding both sides, but use Cauchy–Schwarz*[1] *to get familiar with how it works.)*

(b) Fix $n \geq 1$*. Prove that* $(x_1+\cdots+x_n)^2 \leq n(x_1^2+\cdots+x_n^2)$ *for all* $x_1, \ldots, x_n \in \mathbb{R}$*. Hint: whatever you do, for the love of algebra, do not expand both sides. Instead use Cauchy–Schwarz.*

Solution. □

Problem 2 (Axler 6.B.1). *Fix* $t \in \mathbb{R}$*. Let* $u_t = (\cos t, \sin t)$ *and* $v_t = (-\sin t, \cos t)$ *and* $w_t = (\sin t, -\cos t)$*. Show that* (u_t, v_t) *and* (u_t, w_t) *are each orthonormal bases of* $\mathbb{R}^2$*. Show that every orthonormal basis of* $\mathbb{R}^2$ *has this form, i.e. for any orthonormal basis* z_1, z_2*, there is* $t \in \mathbb{R}$ *so that* (z_1, z_2) *is either equal to* (u_t, v_t) *or* (u_t, w_t)*.*

Solution. □

Problem 3 (Axler 6.A.5). *Suppose* $T \in L(V)$ *is such that* $|Tv| \leq |v|$ *for every* $v \in V$*. Prove that* $T - \sqrt{2}I$ *is invertible.*

Solution. □

[1] On what vector space with what inner product?!

2 For Beckham M.

Problem 4 (Treil 5.3.10). *On $Poly_3(\mathbb{R})$, consider the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$. Apply Gram-Schmidt to the basis $1, x, x^2, x^3$ to produce an orthonormal basis*[2] *for $Poly_3(\mathbb{R})$. Please do this without a computer*[3] *and show your steps.*[4]

Solution. □

Problem 5 (Axler 6.A.7). *Let V be an inner product space. Fix $u, v \in V$. Show that $|au + bv| = |bu + av|$ for all $a, b \in \mathbb{R}$ if and only if $|u| = |v|$.*

Solution. □

Problem 6 (Axler 6.B.10). *Suppose V is a real inner product space and $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Prove that there are exactly 2^m orthonormal lists $e_1, \ldots, e_m$ of vectors such that in V such that*

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for each $j = 1, \ldots, m$.

Solution. □

[2] The polynomials you get are called Legendre polynomials.

[3] If you like, check your answer with Mathematica.

[4] To make this computation easier, it may be helpful to observe that $\int_{-1}^{1} x^{2k+1} = 0$ for any $k \geq 0$ because odd polynomials are odd functions.

3 For Davis L.

Problem 7 (Treil 5.3.8)**.** *Let V be a real inner product space of dimension n. Let $E \subset V$ be subspace of dimension r, and let $P : V \to V$ be the orthogonal projection onto E. Find the eigenvalues and eigenvectors of P, and determine the algebraic and geometric multiplicities of each eigenvalue.*

Solution. □

Problem 8 (Axler 6.B.9)**.** *What happens when Gram–Schmidt is applied to a list of vectors that is not linearly independent?*

Solution. □

Problem 9. *Fix $A \in M_{n\times m}(F)$. Give an alternate proof*[5] *of the rank theorem* $\operatorname{rank}(A) = \operatorname{rank}(A^t)$ *as follows.*

(a) *Show that the span of the rows of A is the orthogonal complement to $\ker A$. Deduce that $m = \dim \operatorname{rank}(A^t) + \dim \ker A$.*

(b) *Use (a) together with the rank-nullity theorem to prove the rank theorem.*

Solution. □

[5] It would be good to go back and remember how we proved the rank theorem before.

4 For Joey F.

Problem 10. *If* $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, *then we can define*

$$\langle \cdot, \cdot \rangle : \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$$

by $\langle v, w \rangle = v^t A w$. *This will always be symmetric and bilinear, but not always positive definite. Show that it's positive definite if and only if* $a, d > 0$ *and* $\det A > 0$. *Hint: complete the square.*

Solution. □

Problem 11 (Axler 6.A.19,21)**.** *Let* V *be a vector space. We say a norm* $|\cdot| : V \to \mathbb{R}$ *satisfies the parallelogram law if*

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

for every $x, y \in V$. *Here you'll prove that if a norm satisfies the parallelogram law, then it comes from an inner product.*[6]

(a) *Assume* V *is an inner product space. Show that*

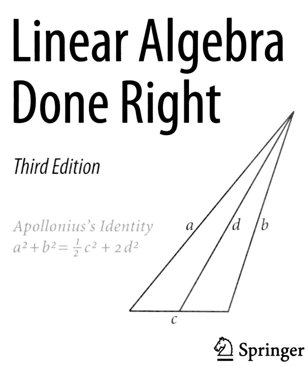
$$\langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4}$$

for all $x, y \in V$.

(b) *Let* $|\cdot| : V \to \mathbb{R}$ *be a norm. Show that if* $|\cdot|$ *satisfies the parallelogram law, then there exists an inner product* $\langle \cdot, \cdot \rangle$ *on* V *so that* $|x| = \langle x, x \rangle^{1/2}$ *for every* $x \in V$. *Hint: first you need to define* $\langle \cdot, \cdot \rangle$; *then you need to check that* $\langle v, v \rangle = |v|^2$ *for every* v; *and then you need to show that* $\langle \cdot, \cdot \rangle$ *is an inner product.*

Solution. □

Problem 12 (Axler 6.A.31)**.** *Use inner products to prove Apollonius's identity: in a triangle with sides of length* a, b, c, *let* d *be the length of the line segment from the midpoint of the side of length* c *to the opposite vertex. Then* $a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$.



Does this look familiar?

[6] We proved the converse in class, so altogether we see that a norm comes from an inner product if and only if it satisfies the parallelogram law.

Solution. □