Homework 7

Math 25a

Due November 2, 2018

Topics covered (lectures 13-14): determinants, multilinear maps

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Davis

Problem 1 (Treil 3.3.2)**.** *How are the determinants of A and B related?*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_2 \end{pmatrix}$$

Solution. □

Problem 2.

(a) *Compute the determinant of* $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ *using row/column operations.*

(b) *Compute the determinant of* $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ *using cofactor expansion.*

Solution. □

Problem 3 (Treil 3.3.10)**.** *Fix $A \in M_n(F)$. Show that the block triangular matrix $\begin{pmatrix} A & * \\ 0 & I_m \end{pmatrix}$ has determinant equal to $\det A$. (Here $I_m \in M_m(F)$ is the $m \times m$ identity matrix.) Use this to show that if $A \in M_n(F)$ and $C \in M_m(F)$, then*

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det C.$$

Solution. □

2 For Joey

Problem 4 (Treil 3.3.4)**.** *$A \in M_n(F)$ is called skew-symmetric if $A^t = -A$. Prove that if A is skew-symmetric and n is odd, then $\det A = 0$. Is this true for n even?*

Solution. □

Problem 5 (Treil 3.4.1)**.** *Suppose the permutation σ takes $(1, 2, 3, 4, 5)$ to $(5, 4, 1, 2, 3)$.*

(a) Find the sign of σ.

(b) Describe the permutation $\sigma^2 = \sigma \circ \sigma$.

(c) Describe the permutation σ^{-1}.

(d) What is the sign of σ^{-1}?

Solution. □

Problem 6 (Treil 3.4.2)**.** *Let P be a permutation matrix, i.e. an $n \times n$ matrix consisting of zeros and ones such that there is exactly one 1 in every row and every column. In the 3×3 case there are 6 permutation matrices; two of them are*

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Give a geometric description for the linear map corresponding to P. This will explain the name "permutation matrix".

(b) Show P is invertible and describe P^{-1}.

(c) Show that there is $N > 0$ so that $P^N = I$. Hint: use the fact that there are only finitely many permutation matrices.

Solution. □

3 For Laura

Problem 7 (Treil 3.4.3). *Why is the number of permutations of $\{1, 2, \ldots, 9\}$ even, and why are exactly half of the permutations odd?*[1] *Hint: this is hard to see in terms of permutations, but easier if you use determinants.*

Solution. □

Problem 8 (Treil 3.4.5 and 3.5.7). *This problem is about the algorithmic complexity of computing determinants. Let A be an $n \times n$ matrix.*

(a) How many multiplications and additions are required to compute $\det A$ *using the formula for the determinant?*

(b) How many multiplications are needed to compute $\det A$ *using cofactor expansion? Explain your answer.*

Solution. □

Problem 9 (Axler 10.B.12). *Let a, b, c be positive numbers. Find the volume of the ellipsoid*

$$\left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\right\}$$

by finding a set $\Omega \subset \mathbb{R}^3$ whose volume you know and an operator $T \in L(\mathbb{R}^3)$ such that $T(\Omega)$ is equal to the ellipsoid above.[2]

Solution. □

[1] N.B. “even” and “odd” have two different sorts of meanings in this sentence!

[2] You may use the formula $\mathrm{vol}(T(\Omega)) = |\det T| \cdot \mathrm{vol}(\Omega)$ from Axler 10.54. This vibes with our intuition that the determinant measures volume expansion of a linear map. We'll prove this formula eventually.

4 For Beckham

Problem 10. *Compute the determinant of a rotation matrix. Does this match your intuition?*

Solution. □

Problem 11 (Treil 3.5.6)**.** *A Vandermonde matrix is a square matrix of the form*

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

Here you'll prove that such a matrix has determinant $\prod_{0 \le i < j \le n}(c_j - c_i)$*.*

(a) *Check that the formula holds for* $n = 1, 2$*.*

(b) *Call the variable* c_n *in the last row* x*, and show that the determinant is a polynomial*

$$P = A_n x^n + \cdots + A_1 x + A_0$$

with the coefficients A_k *depending on* $c_0, \ldots, c_{n-1}$*. (You should not try to find an explicit expression for* A_k *in terms of* $c_0, \ldots, c_{n-1}$*.)*

(c) *Show that* $c_0, c_1, \ldots, c_{n-1}$ *are all roots of* P*, i.e.* $P(c_i) = 0$ *for* $i = 0, \ldots, n-1$*. Conclude*[3] *that* $P = A_n(x - c_0)(x - c_1)\cdots(x - c_{n-1})$*.*

(d) *Assuming the formula for the Vandermonde determinant is ture for* $n - 1$*, compute* A_n *and prove the formula for* n*.*

Solution. □

Problem 12. *Use a Vandermonde matrix to prove: if* $(z_i, w_i) \in \mathbb{C}^2$ *for* $i = 0, \ldots, n$*, and* $z_i \neq z_j$ *for* $i \neq j$*, then there exists a unique polynomial* $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathit{Poly}_n(\mathbb{C})$ *so that* $p(z_i) = w_i$*. Hint: view the problem in terms of a system of equations!*

Solution. □

[3] Use the *Root theorem:* Fix $p \in \text{Poly}(F)$ with degree $n \geq 1$. If $p(c) = 0$, then there exists $q \in \text{Poly}(F)$ with degree $n - 1$ so that $p = q \cdot (x - c)$. (We will prove this theorem eventually.)