Homework 1

Math 25a

Due September 14, 2018 at 5pm

Topics covered (lectures 1-2): sets, functions, cardinality, equivalence relations

Instructions:

- The homework is divided into one part for each CA. You'll submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center. Failure to staple your homework could result in loss of points.
- If you collaborate with other students, please mention this near the corresponding problems.
- For the first two assignments, we would like you to keep track of how long it takes you to complete the entire assignment. Please write that somewhere on your solutions, so we can adjust the difficulty accordingly.
- Some problems from this assignment come from Simmon's book *Introduction to topology and modern analysis*. I've indicated this next to the problems (e.g. Simmons 1.2.3 means problem 3 from the exercises to Section 2 of Chapter 1). You can find a digital copy of Simmons' book on the course website.

1 For Laura

Problem 1. *Let A and B be subsets of a set X. Show $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.*

Solution. □

Problem 2 (Simmons 1.1.3)**.** *(a) The set $X = \{1\}$ has two subsets: the empty set $\emptyset$ and X itself. If A and B are arbitrary subsets of X, then there are four possible relations of the form $A \subset B$. Count the number of true relations among these.*

(b) The set $X = \{1, 2\}$ has four subsets. List them. If A and B are arbitrary subsets of X, then there are 16 possible relations of the form $A \subset B$. Count the number of true relations among these.

(c) The set $X = \{1, 2, 3\}$ has eight subsets. List them. If A and B are arbitrary subsets of X, then there are 64 possible relations of the form $A \subset B$. Count the number of true relations among these.

(d) Let $X = \{1, \ldots, n\}$ for an arbitrary integer n. How many subsets are there? If A and B are arbitrary subsets of X, how many relations of the form $A \subset B$ are there? Based on the evidence you've collected, guess how many relations are true. Prove it.

Solution. □

Problem 3. *Find an explicit formula for the bijection $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined in class. Hint: First give a formula for $\phi(k-1, 1)$, which is the number of points of $\mathbb{N} \times \mathbb{N}$ that lie on the first $(k-1)$ diagonals. Then observe that $\phi(1, k) = \phi(k-1, 1) + 1$ and $\phi(m, k+1-m) = \phi(k-1, 1) + m$. Combine everything to write a formula $\phi(m, n) = \ldots$ given only in terms of m and n.*

Solution. □

2 For Beckham

For functions $f : X \to Y$ and $h : Y \to Z$ defined on sets X, Y, Z, the *composition*, denoted $h \circ f$ or hf, is the function $x \mapsto h(f(x))$. Two mappings $f : X \to Y$ and $g : X \to Y$ are said to be *equal*, denoted $f = g$, if $f(x) = g(x)$ for every $x \in X$. For any set X, the *identity function* $i_X : X \to X$ is defined by $i_X(x) = x$ for every $x \in X$.

Problem 4 (Simmons 1.3.3)**.** *Let X and Y be nonempty sets and $f : X \to Y$ a mapping.*

(a) Show that f is injective if and only if there exists a mapping $g : Y \to X$ so that $gf = i_X$.

(b) Show that f is surjective if and only if there exists a mapping $h : Y \to X$ so that $fh = i_Y$.

Solution. □

Problem 5. *Let $X_1, X_2, \ldots$ be a collection of countable sets. Prove that $\bigcup_{n \in \mathbb{N}} X_n$ is countable.*

Solution. □

Problem 6. *Explain why there are countably many points in $\mathbb{R}^2$ with rational coordinates.*

Solution. □

3 For Davis

Problem 7 (Simmons 1.5.4)**.** *Determine which of the three properties reflexivity, symmetry, transitivity is true for the following relations on the positive integers: m is less than or equal to n ($m \leq n$), m is less than n ($m < n$), m divides n. (You don't need to give proofs.)*

Solution. □

Problem 8 (Simmons 1.5.5)**.** *Give an example of a nonempty set X with a relation $\sim$ that is*

(a) reflexive, but not symmetric or transitive;

(b) symmetric, but not reflexive or transitive;

(c) transitive, but not reflexive or symmetric;

(d) reflexive and symmetric, but not transitive;

(e) reflexive and transitive, but not symmetric;

(f) symmetric and transitive, but not reflexive.

Hint: you can do most of these with X a set of 3 elements.

Solution. □

Problem 9. *Let $f : X \to Y$ be any map. Define an relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Prove this is an equivalence relation.*

Solution. □

4 For Joey

Problem 10 (Fundamental theorem of caveman math). *Let $m, n \geq 1$ be integers. The point of this problem is to prove that if the sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$ have the same cardinality, then $m = n$. For brevity, denote $\{1, \dots, n\}$ by $[n]$. Show each of the following.*

(a) If $f : [m] \to \{1\}$ is a bijection, then $m = 1$.

(b) Fix $k \in [n]$. Define a bijection $g : [n] \to [n]$ such that $g(k) = n$ and $g(n) = k$.

(c) If $f : [m] \to [n]$ is a bijection, then there exists another bijection $f' : [m] \to [n]$ so that $f'(m) = n$.

(d) If $f : [m] \to [n]$ is a bijection and $f(m) = n$, then the restriction of f to $[m-1]$ defines a bijection $[m-1] \to [n-1]$.

Use induction on n (and the above) to prove that card$([m])$ = card$([n])$ implies $m = n$.

Solution. □

Problem 11. *Prove Cantor's theorem: for any set X, there is no surjection $f : X \to P(X)$. Hint: try to mimic the argument of Russell's paradox (the proof is not long).*

Solution. □

Problem 12 (Olbers' paradox, circa 1823). *In class we viewed the positive rational numbers as a subset of the infinite grid $\mathbb{N} \times \mathbb{N}$ in $\mathbb{R}^2$. This grid plays an interesting role in cosmology.*

(a) Suppose that each point of $\mathbb{N} \times \mathbb{N}$ is surrounded by a disk of a fixed radius $r > 0$. Show that every line through 0 intersects one of these disks. Hint: Consider the slope α of the line. The case α is rational is easier. If α is irrational, use the pigeonhole principle[1] to conclude that for any $r > 0$ there is n, m in $\mathbb{N}$ so that the fractional parts[2] of $n\alpha$ and $m\alpha$ are within r.

(b) Conclude that if space were uniformly filled with stars, the whole sky would be filled with light! This is called Olbers' paradox.

Solution. □

[1] If you put pigeons in holes and there are fewer holes than pigeons, then one hole has more than one pigeon in it. This was also known to cavemen.

[2] Every positive number x can be written uniquely as $x = n + \{x\}$ with $n \in \mathbb{N}$ and the *fractional part* $\{x\}$ in $[0, 1)$.

5 LaTeX Guide

If you want to define a piecewise function, here's one way to do it:

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

When you write out a string of equations, sometimes, it's nice to list the equations horizontally, as follows:

$$\begin{aligned} (x+y)(z+w) &= (x+y)z + (x+y)w \\ &= (xz+yz) + (xw+yw) \\ &= xz+yz+xw+yw \end{aligned}$$

The number of subsets of size k of a set of size n is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. These numbers are also known as the binomial coefficients because

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This is known as the binomial theorem.

You can make tables in LaTeX:

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}