# Homework 5

Math 25a

Due October 11, 2017

Topics covered: matrices, vector space of linear maps, isomorphisms, quotient spaces

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

# 1 For Natalia

**Problem 1** (Axler 3.C.2)**.** *Let* $D : \mathrm{Poly}_3(\mathbb{R}) \to \mathrm{Poly}_2(\mathbb{R})$ *be the differentiation map. Find a basis of* $\mathrm{Poly}_3(\mathbb{R})$ *and a basis of* $\mathrm{Poly}_3(\mathbb{R})$ *such that the matrix of* $D$ *with respect to these bases is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

*Solution.* □

**Problem 2** (Axler 3.B.5-6)**.** *(a) Give an example of a linear map* $T : \mathbb{R}^4 \to \mathbb{R}^4$ *such that* $\operatorname{Im} T = \ker T$.

*(b) Prove that there is no linear map* $T : \mathbb{R}^5 \to \mathbb{R}^5$ *such that* $\operatorname{Im} T = \ker T$.

*Solution.* □

**Problem 3.** *Let* $V$ *and* $W$ *be finite dimensional vector spaces over* $F$ *with bases* $(v_1, \ldots, v_n)$ *and* $(w_1, \ldots, w_m)$ *respectively.*

*(a) Define a map* $T_{ij} : V \to W$ *that sends the* $v_j$ *to* $w_i$ *and maps all the other basis vectors* $v_k$ *to* $0 \in W$*. Why is* $T_{ij}$ *is linear? Describe the matrix for* $T_{ij}$*.*

*(b) Show that the maps* $\{T_{ij} : 1 \le j \le n, 1 \le i \le m\}$ *is a basis for* $L(V, W)$ *and give a formula for the dimension of* $L(V, W)$ *in terms of* $\dim V$ *and* $\dim W$*.*

*(c) If* $W = F$*, take* $w_1 = 1$*, and show that the map taking* $v_j$ *to* $T_{1j}$ *defines a linear isomorphism* $\phi : V \to L(V, F)$*.*

*Solution.* □

## 2 For Michele

**Problem 4** (Axler 3.D.7)**.** *Suppose $V$ and $W$ are finite dimensional. Fix $v \in V$ and let*

$$E = \{T \in L(V, W) : Tv = 0\}$$

*(a) Show that $E$ is a subspace of $L(V, W)$.*

*(b) What is $E$ in the case $v = 0$? Assuming $v \neq 0$, use the rank-nullity theorem to compute the dimension of $E$?*

*Solution.* □

**Problem 5** (Axler 3.B.7)**.** *Suppose $V$ and $W$ are finite dimensional and $2 \leq \dim V \leq \dim W$. Show that $X = \{T \in L(V, W) : T \text{ is not injective}\}$ is not a subspace of $L(V, W)$.*

*Solution.* □

**Problem 6** (Axler 3.B.30)**.** *Suppose that $S, T : V \to F$ are linear maps so that $\ker S = \ker T$. Prove that there exists a constant $c \in F$ so that $T = cS$.*

*Solution.* □

## 3 For Charlie

**Problem 7.** *If $V$ is a vector space and $W \subset V$ is a subspace, the quotient space $V/W$ is defined as the set of equivalence classes of $V$ with respect to the equivalence relation $v_1 \sim v_2$ if $v_1 - v_2 \in W$. As usual, we denote the equivalence class of $v \in V$ by $[v]$. In this exercise, you'll show that $V/W$ is a vector space.*

*(a) Show that the addition and scalar multiplications defined by*

$$[v_1] + [v_2] = [v_1 + v_2] \quad \textit{and} \quad a[v] = [av]$$

*are well-defined on $V/W$.*

*(b) Convince yourself that the addition on $V/W$ is associative and commutative. What is the additive identity? What is the additive inverse of $[v]$? Convince yourself that scalar multiplication is associative, distributive, and $1 \cdot [v] = [v]$ for every $v$.*

*Solution.* □

**Problem 8.** *Consider $V = \mathrm{Poly}_4(\mathbb{C})$ and let $U = \ker f$, where $f : V \to \mathbb{C}$ is defined as $f(p) = p(i)$. Which of the following vectors are equal in $V/U$?*

$$[0], \quad [x+i], \quad [x^4 + 2x^2 + x + 1 + i], \quad [x^2+1], \quad [x].$$

*Solution.* □

**Problem 9.** *Suppose $U \subset V$ is a subspace such that $V/U$ is finite dimensional. Prove that $V$ is isomorphic to $U \times (V/U)$. (Here $U \times (V/U)$ is the product vector space with addition and scalar multiplication defined coordinate-wise. Hint: to show they are isomorphic, you need to define a map $f : V \to U \times (V/U)$ and prove that it is a linear isomorphism.)*

*Solution.* □

## 4 For Ellen

**Problem 10** (Axler 3.E.14). *Consider* $U = \{(x_1, x_2, \dots) \in F^\infty : x_j \neq 0$ *for only finitely many* $j\}$.

*(a) Show that* $U$ *is a subspace of* $F^\infty$.

*(b) Prove that* $F^\infty/U$ *is infinite dimensional.*

*Solution.* □

**Problem 11** (Axler 3.D.1). *Suppose that* $T : U \to V$ *and* $S : V \to W$ *are invertible linear maps. Prove that* $ST : U \to W$ *is invertible with inverse* $(ST)^{-1} = T^{-1}S^{-1}$.

*Solution.* □

**Problem 12** (Axler 3.D.19). *Let* $V = \mathrm{Poly}(\mathbb{R})$. *Suppose that* $T : V \to V$ *is injective and* $\deg Tp \leq \deg p$ *for every nonzero polynomial* $p \in V$.

*(a) Prove that* $T$ *is surjective.*

*(b) Prove that* $\deg Tp = \deg p$ *for every nonzero* $p \in V$.

*Solution.* □