Homework 1

Math 25a

Due September 6, 2017

Topics covered: sets, functions, cardinality

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center. Failure to staple your homework could result in loss of points.
- If you collaborate with other students, please mention this near the corresponding problems.
- For the first two assignments, we would like you to keep track of how long it takes you to complete the entire assignment. Please write that somewhere on your solutions, so we can adjust the difficulty accordingly.
- Some problems from this assignment come from Simmon's book *Introduction to topology and modern analysis*. I've indicated this next to the problems (e.g. Simmons 1.2.3 means problem 3 from the exercises to Section 2 of Chapter 1). You can find a digital copy of Simmons' book on the course website.

1 For Charlie

Problem 1. *Let A and B be subsets of a set X. Show that $(A \cup B)^c = A^c \cap B^c$. What is $(A \cap B)^c$?*

Solution. Put your solution here. □

Problem 2. *Fix a set X. The* difference *between two subsets $A, B \subset X$, denoted $A - B$, is by definition*

$$A - B = \{x \in X : x \in A \text{ and } x \notin B\}.$$

Equivalently, $A - B = A \cap B^c$. Prove that $A - (A \cap B) = A - B$ and $(A \cup B) - B = A - B$.

Solution. □

Problem 3 (Simmons 1.1.3)**.** *(a) The set $X = \{1\}$ has two subsets, namely the empty set $\emptyset$ and X itself. If A and B are arbitrary subsets of X, then there are four possible relations of the form $A \subset B$. Count the number of true relations among these.*

(b) The set $X = \{1, 2\}$ has four subsets. List them. If A and B are arbitrary subsets of X, then there are 16 possible relations of the form $A \subset B$. Count the number of true relations among these.

(c) The set $X = \{1, 2, 3\}$ has eight subsets. List them. If A and B are arbitrary subsets of X, then there are 64 possible relations of the form $A \subset B$. Count the number of true relations among these.

(d) Let $X = \{1, \ldots, n\}$ for an arbitrary integer n. How many subsets are there? If A and B are arbitrary subsets of X, how many relations of the form $A \subset B$ are there? Based on the evidence you've collected, guess how many relations are true. Try to prove it (this may be challenging).

Solution. □

Problem 4. *In the first lecture we discussed* Russell's paradox*, which involved the "set"*

$$N = \{X : X \notin X\}$$

of all normal sets. Briefly explain Russell's paradox by supposing that N is either normal or abnormal and explaining what those assumptions would imply. (Make sure to justify each deduction. Your explanation should have more words than symbols.)

Solution. □

2 For Michele

For functions $f : X \to Y$ and $h : Y \to Z$ defined on sets X, Y, Z, the *composition*, denoted $h \circ f$ or hf, is the function defined by $(hf)(x) = h(f(x))$. Two mappings $f : X \to Y$ and $g : X \to Y$ are said to be *equal*, denoted $f = g$, if $f(x) = g(x)$ for every $x \in X$.

Problem 5 (Simmons 1.3.1)**.** *Let f, g, h be any three mappings of a nonempty set X to itself. Show that composition of mappings is associative, i.e. $f(gh) = (fg)h$.*

Solution. □

Problem 6 (Simmons 1.3.3)**.** *Let X and Y be nonempty sets and $f : X \to Y$ a mapping.*

(a) Show that f is injective if and only if there exists a mapping $g : Y \to X$ so that $gf = i_X$.

(b) Show that f is surjective if and only if there exists a mapping $h : Y \to X$ so that $fh = i_Y$.

Solution. □

Problem 7. *Consider two nonempty sets X and Y and their product set*

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

Consider the "projection" maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ defined by $p_X(x, y) = x$ and $p_Y(x, y) = y$. These maps are surjective (why?). For a mapping $f : X \to Y$ define the graph

$$G_f = \{(x, f(x)) : x \in X\} \subset X \times Y.$$

When restricted to G_f, when are the projection maps bijections? (You may find the previous problem useful.)

Solution. □

3 For Ellen

An *abelian group* is a set A with an operation $\star : A \times A \to A$ that satisfies the following properties:

1. (Commutativity) $a \star b = b \star a$ for every $a, b \in A$.
2. (Associativity) $(a \star b) \star c = a \star (b \star c)$ for every $a, b, c \in A$.
3. (Identity) There exists an element $e \in A$ so that $a \star e = a$ for every $a \in A$.
4. (Inverses) For each $a \in A$, there is an element $a' \in A$ so that $a \star a' = e$.

For example, the integers $\mathbb{Z}$ with addition as the operation is an abelian group. On the other hand, $\mathbb{Z}$ under multiplication and the nonnegative integers $\{0, 1, 2, \ldots\}$ under addition are not abelian groups because not every element has an inverse.

Problem 8. *Let X be a nonempty set and let $P(X)$ be the power set. Which of the abelian group axioms hold for the union operation $\cup : P(X) \times P(X) \to P(X)$.*

Solution. □

Problem 9. *Let X be a set. For subsets $A \subset X$ and $B \subset X$ the* symmetric difference *is defined as*

$$A \Delta B = (A - B) \cup (B - A).$$

We can view this as an operation $\Delta : P(X) \times P(X) \to P(X)$. Which of the abelian group axioms does this operation satisfy? (Hint: for associativity, draw a picture first.)

Solution. □

Problem 10. *Let $m, n \geq 1$ be integers. The point of this problem is to prove that if the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ have the same cardinality, then $m = n$. For brevity, denote $\{1, \ldots, n\}$ by $[n]$. Show each of the following claims.*

(a) If $f : [m] \to \{1\}$ is a bijection, then $m = 1$.

(b) Given any $k \in [n]$, there exists a bijection $g : [n] \to [n]$ so that $g(k) = n$ and $g(n) = k$.

(c) If $f : [m] \to [n]$ is a bijection, then there exists another bijection $f' : [m] \to [n]$ so that $f'(m) = n$.

(d) If $f : [m] \to [n]$ is a bijection and $f(m) = n$, then the restriction of f to $[m-1]$ defines a bijection $[m-1] \to [n-1]$.

Explain how to use the above claims to prove that card$([m])$ = card$([n])$ implies $m = n$.

Solution. □

4 LaTeX Guide

If you want to define a piecewise function, here's one way to do it:

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

When you write out a string of equations, sometimes, it's nice to list the equations horizontally, as follows:

$$\begin{aligned} (x+y)(z+w) &= (x+y)z + (x+y)w \\ &= (xz+yz) + (xw+yw) \\ &= xz + yz + xw + yw \end{aligned}$$

The number of subsets of size k of a set of size n is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. These numbers are also known as the binomial coefficients because

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

A basic relation satisfied by binomial coefficients is

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

(If you want to choose k elements from $\{a_1, \ldots, a_{n+1}\}$, you can either choose k elements of $\{a_1, \ldots, a_n\}$ or you could choose a_{n+1} together with $k-1$ elements of $\{a_1, \ldots, a_n\}$.) Sometimes this equality is useful in inductive proofs.